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Semiclassical analysis of Wigner 3*j*-symbol

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Abstract

We analyse the asymptotics of the Wigner $3j$-symbol as a matrix element connecting eigenfunctions of a pair of integrable systems, obtained by lifting the problem of the addition of angular momenta into the space of Schwinger's oscillators. A novel element is the appearance of compact Lagrangian manifolds that are not tori, due to the fact that the observables defining the quantum states are noncommuting. These manifolds can be quantized by generalized Bohr–Sommerfeld rules and yield all the correct quantum numbers. The geometry of the classical angular momentum vectors emerges in a clear manner. Efficient methods for computing amplitude determinants in terms of Poisson brackets are developed and illustrated.

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1. Introduction

This paper is a study of the asymptotics of the Wigner $3j$-symbol from the standpoint of semiclassical mechanics, that is, essentially multidimensional WKB theory for integrable systems. The principal result itself, the leading asymptotic expression for the $3j$-symbol, has been known since Ponzano and Regge (1968). Nevertheless our analysis presents several novel features. One is the exploration of Lagrangian manifolds in phase space that are not tori (the usual case for eigenstates of integrable systems). Instead, one of the states entering into the $3j$-symbol is supported semiclassically on a Lagrangian manifold that is a nontrivial 3-torus bundle over $SO(3)$. This manifold can be quantized by generalized Bohr–Sommerfeld rules, whereupon it yields the exact eigenvalues required by the quantum $3j$-symbol, as well as the correct amplitude and phase of its asymptotic form. This unusual Lagrangian manifold arises because the quantum state in question is an eigenstate of a set of noncommuting operators. Other novel features include the expression of the asymptotic phase of the $3j$-symbol in terms of the phases of Schwinger's harmonic oscillators and the determination of

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stationary phase points by geometrically transparent operations on angular momentum vectors in three-dimensional space. Yet another is the representation of multidimensional amplitude determinants as matrices of Poisson brackets. Representations of this type have been known for some time, but they are generalized here to the case of sets of noncommuting operators. The final result is a one-line derivation of the amplitude of the asymptotic form of the $3j$-symbol. Similarly, brief derivations are possible for the amplitudes of the $6j$- and $9j$-symbols.

In addition, our analysis of the $3j$-symbol may prove to be useful for the asymptotic study of the $3nj$-symbols for higher n. The leading order asymptotics of the $6j$-symbol were derived by Ponzano and Regge (1968), but the understanding of the asymptotics of the $9j$-symbol is still incomplete. These symbols are important in many applications in atomic, molecular and nuclear physics, for example, the $9j$-symbols are needed in atomic physics to convert from an LS-coupled basis to a jj-coupled basis. These symbols are all examples of closed spin networks, of which more elaborate examples occur in applications, each of which presents a challenge to asymptotic analysis. Moreover, in recent years new interest in this subject has arisen from researches into quantum computing (Marzuoli and Rasetti 2005) and quantum gravity, where new derivations of the asymptotics of the Wigner $6j$-symbol have been produced as well as generalizations to other groups such as the Lorentz group. The $3nj$-symbols and their asymptotics have also been used recently in algorithms for molecular quantum mechanics (De Fazio *et al* 2003, Anderson and Aquilanti 2006), which exploit the connections with the theory of discrete orthogonal polynomials (Aquilanti *et al* 1995, 2001a, 2001b and references therein).

The asymptotic formula for the $3j$-symbol is closely related to that for the $6j$-symbol, being a limiting case of the latter. These were first derived by Ponzano and Regge (1968), using intuitive methods and building on Wigner's earlier result for the amplitude of the $6j$-symbol (Wigner 1959). Later Neville (1971) analysed the asymptotics of the $3j$- and $6j$- symbols by a discrete version of WKB theory, applied to the recursion relations satisfied by those symbols, without apparently knowing of the work of Ponzano and Regge. His formulae are not presented in a particularly transparent or geometrical manner, but appear to reproduce some of the results of Ponzano and Regge. The formula for the $3j$-symbol (in the form of Clebsch–Gordan coefficients) was later derived again by Miller (1974), who presented it as an example of his general theory of semiclassical matrix elements of integrable systems. Miller called on the fact that the phase of the semiclassical matrix element is a generating function of a canonical transformation and used the classical transformation that most obviously corresponds to the quantum addition of angular momenta to reconstruct the generating function. The method leads to a difficult integral, which, once done, yields the five terms in the phase of the asymptotic formula for the $3j$-symbol. Somewhat later Schulten and Gordon (1975a, 1975b) presented a rigorous derivation of the Ponzano and Regge results for the $3j$- and $6j$-symbols, using methods similar to those of Neville but carrying them out in a more thorough and elegant manner. Schulten and Gordon also provided uniform approximations for the transition from the classical to nonclassical regimes, work that has recently been reanalysed (Geronimo *et al* 2004) and extended to non-Euclidean and quantum groups (Taylor and Woodward 2005). Somewhat later Biedenharn and Louck (1981b) presented a review and commentary of the results of Ponzano and Regge, as well as a proof based on showing that the result satisfies asymptotically a set of defining relations for the $6j$-symbol. More recently, the asymptotics of the $3j$-symbol was derived again by Reinsch and Morehead (1999), working with an integral representation constructed out of Wigner's single-index sum for the Clebsch–Gordan coefficients. About the same time, Roberts (1999) derived the Ponzano and Regge results for the $6j$-symbol, using methods of geometric quantization. Finally, Freidel and Louapre (2003) presented a derivation of the asymptotic expression for the square of the $6j$-symbol, based on an analysis of an $SU(2)$ path integral. This work was part of a larger study of generalizations of the $6j$-symbol to other groups (for example, the $10j$-symbol) that are important in quantum gravity. See also Barrett and Steele (2003) and Baez, Christensen and Egan (2002).

There are many variations on the calculation of the asymptotic forms of the $3nj$-symbols that have been considered by different authors. There are asymptotic forms inside and outside the classically allowed regions, uniform approximations connecting two or more of these regions, asymptotic forms when only some of the quantum numbers are large and others small, and higher order terms. Ponzano and Regge (1968) covered many of these issues, while Reinsch and Morehead computed some higher order terms.

The outline of this paper is as follows. In section 2, we review the semiclassical mechanics of integrable systems in the generic case that one has sets of commuting observables, drawing attention to an expression for the amplitude determinant in terms of Poisson brackets. In section 3, we review the Schwinger model for representing angular momentum operators in terms of harmonic oscillators. This model allows us to express angular momentum eigenstates in terms of wavefunctions on $\mathbb{R}^n$, which we use in section 4 to express the $3j$-symbols in terms of scalar products of such functions. In section 5, we study the Schwinger model from a classical standpoint, in which an important element is the reduction of the Schwinger phase space (the 'large phase space') by the torus group T^3, producing the Poisson manifold $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ ('angular momentum space') and the reduced phase space $S^2 \times S^2 \times S^2$ (the 'small phase space'). In sections 6 and 7, we study the two Lagrangian manifolds that support the states whose scalar product is the $3j$-symbol. One is a conventional invariant torus (the 'jm-torus'), but the other, what we call the 'Wigner manifold', is compact and Lagrangian but not a torus. This manifold supports Wigner's state of zero total angular momentum that enters into the definition of the $3j$-symbols. In sections 8, we study the intersections of the jm-torus and the Wigner manifold, which are the stationary phase points of the $3j$-symbol, and show how these can be found by elementary geometrical considerations in three-dimensional space (that is, by rotating angular momentum vectors). The intersection of the two manifolds turns out to be a pair of 4-tori. In section 9, we compute the action integrals along the respective Lagrangian manifolds to points on the two 4-tori, whose difference is the Ponzano and Regge phase of the $3j$-symbol. In section 10, we apply generalized Bohr–Sommerfeld quantization to the jm-torus and the Wigner manifold, a standard procedure for the jm-torus, although it leads in an interesting way to the extra $1/2$ in the classical values representing the lengths of the angular momentum vectors. This extra $1/2$ was guessed by Ponzano, Regge and Miller and derived systematically by Schulten and Gordon, Reinsch and Morehead, Roberts and by us, although it enters somewhat asymmetrically in the work of Roberts. In our work, it is essentially a Maslov index. In section 11, we generalize known expressions for the amplitude determinant of semiclassical matrix elements of integrable systems in terms of Poisson brackets to the case of collections of noncommuting observables (whose level sets nevertheless are Lagrangian). The result allows us to compute the amplitude of the $3j$-symbol as a 2×2 matrix of Poisson brackets. We then put all the pieces together to obtain the final asymptotic form. Finally, in section 12, we present some comments on the work, prospects for further work and conclusions.

2. Semiclassical wavefunctions for integrable systems

The semiclassical mechanics of integrable systems is well understood (Einstein 1917, Brillouin 1926, Keller 1958, Percival 1973, Berry and Tabor 1976, Gutzwiller 1990, Brack and Bhaduri 1997, Cargo *et al* 2005a, 2005b). Here, we summarize the basic facts, some of which require modification for our application.

We consider the quantum mechanics of a particle moving in $\mathbb{R}^n$ (with wavefunction $\psi(x_1, \ldots, x_n)$ and Hilbert space $L^2(\mathbb{R}^n)$). We speak of an integrable system if we have a complete set of commuting observables $\{\hat{A}_1, \ldots, \hat{A}_n\}$ acting on this Hilbert space. We use hats to distinguish quantum operators from classical quantities with a similar meaning. Sometimes the Hamiltonian is one of these operators or a function of them, but in our application there is no Hamiltonian, or rather all $\hat{A}_i$'s are Hamiltonians on an equal footing. These operators may be converted into their classical counterparts by the Weyl transform (Weyl 1927, Wigner 1932, Groenewold 1946, Moyal 1949, Voros 1977, Berry 1977, Balazs and Jennings 1984, Hillery *et al* 1984, Littlejohn 1986, McDonald 1988, Estrada *et al* 1989, Gracia-Bondía and Várilly 1995, Ozorio de Almeida 1998). The Weyl transforms (or Weyl 'symbols') of these operators are functions on the classical phase space $\mathbb{R}^{2n}$, that is, functions of $(x_1, \ldots, x_n; p_1, \ldots, p_n)$. They are normally even power series in $\hbar$, as we assume, of which the leading term is the 'principal symbol'. We denote the principal symbols of $\{\hat{A}_1, \ldots, \hat{A}_n\}$ by $\{A_1, \ldots, A_n\}$ (without the hats). In view of the Moyal star product representation (Moyal 1949) of the vanishing commutators $[\hat{A}_i, \hat{A}_j] = 0$, the principal symbols Poisson commute, $\{A_i, A_j\} = 0$, thus defining a classically integrable system (Arnold 1989, Cushman and Bates 1997). (We use curly brackets {} both to denote a set and for Poisson brackets.) Then according to the Liouville–Arnold theorem (Arnold 1989), the compact level sets of $\{A_1, \ldots, A_n\}$ are generically n-tori. The Hamiltonian vector fields generated by A_i are commuting and linearly independent of the tori; thus the tori are not only the level sets of A_i, they are also the orbits of the Abelian group generated by the corresponding Hamiltonian flows. One can define an action function S on a torus as the integral of $\sum_i p_i\, dx_i$ relative to some initial point; it is multivalued because of the topologically distinct paths going from the initial to the final point, but otherwise is independent of the path.

Let $A_i = a_i$ be one of these tori (A_i are the functions, a_i the values). The torus has a projection onto configuration space defining a classically allowed region in that space; the inverse projection is multivalued. The function S may be projected onto configuration space, defining a function we shall denote by $S_k(x, a)$ (where for brevity x and a stand for $(x_1, \ldots, x_n)$ and $(a_1, \ldots, a_n)$, etc). Here k labels the branches of the inverse projection; function S_k has an additional multivaluedness due to the choice of contour connecting initial and final points on the torus. Then as explained by Arnold (1989), $S(x, a)$ is the generating function of the canonical transformation $(x, p) \to (\alpha, A)$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is the set of angle variable conjugate to the conserved quantities $(A_1, \ldots, A_n)$. Action variables may be defined in the usual way as $(1/2\pi) \oint p\, dx$ around the independent basis contours on the torus; these are functions of A_i and their conjugate variables are the angles that cover the torus once when varying between 0 and 2π. Sometimes however it is more convenient to work with A_i instead of the actions (A_i are not necessarily actions, and their flows are not necessarily periodic on the torus).

Tori are quantized, that is, associated with a consistent solution of the simultaneous Hamiltonian–Jacobi and amplitude transport equations for the operators $\hat{A}_i$, only if they satisfy the Bohr–Sommerfeld or EBK quantization conditions, discussed in section 10. Associated with a quantized torus is a semiclassical wavefunction in configuration space, which in the classically allowed region is given by

$$\psi(x) = \langle x|a\rangle = \frac{1}{\sqrt{V}} \sum_k |\Omega_k|^{1/2} \exp\{i[S_k(x, a) - \mu_k \pi/2]\}. \tag{1}$$

The meaning of this formula is the following. First, here and below we set $\hbar = 1$. Next, given the point x in the classically allowed region, its inverse projection onto the quantized torus is a set of points indexed by k. We assume that the projection is nonsingular at these points (we are not at a caustic). The phase $S_k(x, a)$ is the integral of $p\,\mathrm{d}x$ from a given initial point on the torus to the kth point of the inverse projection, and μ_k is the Maslov index (Maslov 1981, Mishchenko *et al* 1990, de Gosson 1997) of the same path. The amplitude determinant Ω_k is given by

$$\Omega_k = \det \frac{\partial^2 S_k(x, a)}{\partial x_i \partial a_j} = [\det\{x_i, A_j\}]^{-1}, \tag{2}$$

where in the second form the Poisson brackets are evaluated on the kth branch of the inverse mapping from x to the Lagrangian manifold. The amplitude determinant is a density on configuration space (to within the semiclassical approximation, the probability density corresponding to a single branch), which is the projection onto configuration space of a density on the torus. The latter density is required to be invariant under the Hamiltonian flows generated by A_i (this is the meaning of the amplitude transport equations for A_i); in terms of the variables α_i conjugate to A_i this means that the density is constant (it is the n-form $\mathrm{d}\alpha_1 \wedge \cdots \wedge \mathrm{d}\alpha_n$). Finally, the quantity V in (1) is the volume of the torus, measured with respect to this density. If A_i are action variables, then $V = (2\pi)^n$. The overall phase of the wavefunction (its phase convention) is determined by the choice of the initial point on the torus.

Now let $\{\hat{A}_1, \ldots, \hat{A}_n\}$ and $\{\hat{B}_1, \ldots, \hat{B}_n\}$ be two complete sets of commuting observables, with principal symbols A_i and B_i, conjugate angles α_i and β_i and action functions $S_A(x, a)$ and $S_B(x, b)$, and let a and b refer to two quantized tori (an A-torus and a B-torus). We assume initially that the two sets of functions A_i and B_i are independent. We compute $\langle b|a\rangle$ as an integral of the wavefunctions over x, evaluated by the stationary phase approximation. The stationary phase points are geometrically the intersections of the A-torus with the B-torus. Generically, the two tori intersect in finite set of isolated points that we index by k, denoting the corresponding α and β values by α_k and β_k. (For given k, α_k and β_k refer to the same point in phase space.) Then, the result is

$$\langle b|a\rangle = \frac{(2\pi\,\mathrm{i})^{n/2}}{\sqrt{V_A V_B}} \sum_k |\Omega_k|^{1/2} \exp\{\mathrm{i}[S_A(\alpha_k) - S_B(\beta_k) - \mu_k\pi/2]\}. \tag{3}$$

Here, V_A and V_B are the volumes of the respective tori, as in (1), and the actions S_A and S_B are considered functions of the α or β coordinates on the respective tori.

As shown by Littlejohn (1990), the amplitude determinant Ω_k can be written in terms of the Poisson brackets of the observables A_i and B_i,

$$\Omega_k = [\det\{A_i, B_j\}]^{-1}. \tag{4}$$

The Maslov index μ_k in (3) is not the same as in (1).

Another case considered by Littlejohn (1990) is the one in which some of A_i are functionally dependent on some of B_i. For this case, it is convenient to assume that the first r of the two sets of variables A and B are functionally independent, while the last $n - r$ are identical, so that $A = \{A_1, \ldots, A_r, A_{r+1}, \ldots, A_n\}$ and $B = \{B_1, \ldots, B_r, A_{r+1}, \ldots, A_n\}$. Then the stationary phase points are still the intersections of the two n-tori, but now the intersections are generically a finite set of isolated $(n-r)$-tori, upon which linearly independent vector fields are the Hamiltonian vector fields associated with the $(A_{r+1}, \ldots, A_n)$. Such an $(n - r)$-torus is the orbit of the Abelian group action generated by the corresponding Hamiltonian flows. In this case, we find

$$\langle b|a\rangle = \frac{(2\pi\,\mathrm{i})^{r/2}}{\sqrt{V_A V_B}} \sum_k V_k |\Omega_k|^{1/2} \exp\{\mathrm{i}[S_A(\alpha_k) - S_B(\beta_k) - \mu_k\pi/2]\}, \tag{5}$$

where now V_k is the volume of the kth intersection (an $(n-r)$-torus on which the volume measure is $\mathrm{d}\alpha_{r+1} \wedge \cdots \wedge \mathrm{d}\alpha_n$), and where now the amplitude determinant Ω_k is still given by (4), except that it is understood that only the first r of A's and B's enter (thus, it is an $r \times r$ determinant instead of an $n \times n$ one). The phase difference $S_A - S_B$ for branch k can be evaluated at any point on the $(n-r)$-torus which is the intersection, since the integral of $p\,\mathrm{d}x$ back and forth along a path lying in the intersection vanishes.

3. The Schwinger model

The Schwinger (or $SU(2)$ or boson) model for angular momenta is explained well in Schwinger's original paper (reprinted by Biedenharn and van Dam (1965), the original 1952 paper being unpublished) and reworked in an interesting way by Bargmann (1962). For further perspective see Biedenharn and Louck (1981a) and Smorodinskii and Shelepin (1972). Introductions are given by Sakurai (1994) and Schulman (1981). Here we define the notation for the Schwinger model and emphasize some aspects that will be important for our application.

In the Schwinger model each independent angular momentum vector is associated with two harmonic oscillators. We shall refer to the $1j$-, $3j$-, etc, models, depending on how many independent angular momenta there are. The number of j's in the model is not necessarily the number of j's in the Wigner symbol; for example, Miller (1974) used a $2j$-model to study the Clebsch–Gordan coefficients, essentially the $3j$-symbols.

We start with the $1j$-model, for which there are two harmonic oscillators indexed by Greek indices $\mu, \nu, \ldots = 1, 2$. (These are just labels of the two oscillators; sometimes other labels such as $1/2, -1/2$ are more suitable.) The wavefunctions are $\psi(x_1, x_2)$ and the Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^2)$. We write $\hat{H}_\mu = (1/2)\left(\hat{x}_\mu^2 + \hat{p}_\mu^2\right)$ for the two oscillator Hamiltonians, and we define $\hat{H} = \sum_\mu \hat{H}_\mu$. The eigenvalues of $\hat{H}$ are $n+1$, with $n = 0, 1, \ldots$, and energy level E_n is $(n+1)$-fold degenerate. We introduce usual annihilation and creation operators $a_\mu = (\hat{x}_\mu + \mathrm{i}\hat{p}_\mu)/\sqrt{2}$, $a_\mu^\dagger = (\hat{x}_\mu - \mathrm{i}\hat{p}_\mu)/\sqrt{2}$, omitting the hats on a's and $a^\dagger$'s since these will always be understood to be operators. We define operators

$$\hat{I} = \frac{1}{2}\sum_\mu a_\mu^\dagger a_\mu = \frac{1}{2}(\hat{H} - 1) \tag{6}$$

and

$$\hat{J}_i = \frac{1}{2}\sum_{\mu\nu} a_\mu^\dagger \sigma_{\mu\nu}^i a_\nu, \tag{7}$$

where σ^i is the ith Pauli matrix. Here and below we use indices $i, j, \ldots = 1, 2, 3$ (or x, y, z if that is more clear) to denote the Cartesian components of a 3-vector. Note that $\hat{I}$ and $\hat{J}_i$ are quadratic functions of x's and p's of the system. The eigenvalues of $\hat{I}$ are $n/2$ for $n = 0, 1, \ldots$. These operators satisfy the commutation relations $[\hat{I}, \hat{J}_i] = 0$ and $[\hat{J}_i, \hat{J}_j] = \mathrm{i}\sum_k \epsilon_{ijk}\hat{J}_k$. We also define $\hat{\mathbf{J}}^2 = \sum_i \hat{J}_i^2$, so that $[\hat{I}, \hat{\mathbf{J}}^2] = 0$ and $[\hat{J}_i, \hat{\mathbf{J}}^2] = 0$. It avoids some confusion with indices to always denote the square of a vector by a bold face symbol, as we have done here. We note the important operator identity $\hat{\mathbf{J}}^2 = \hat{I}(\hat{I}+1)$, expressing the quartic operator $\hat{\mathbf{J}}^2$ as a function of the quadratic operator $\hat{I}$.

From this identity and the known eigenvalues of $\hat{I}$ it follows that the eigenvalues of $\hat{\mathbf{J}}^2$ are $(n/2)[(n/2)+1]$, for $n = 0, 1, \ldots$, which leads us to identify $n/2$ with $j = 0, 1/2, 1, \ldots$, the usual angular momentum quantum number. The nth (or jth) eigenspace of $\hat{H}$ or $\hat{I}$ is $(2j+1)$-dimensional, and so must contain a single copy of the jth irrep of $SU(2)$. Each irrep (both integer and half-integer values of j) occurs precisely once in the Hilbert space $\mathcal{H}$.

We denote these subspaces by $\mathcal{H}_j$, and write $\mathcal{H} = \sum_j \oplus \mathcal{H}_j$. The standard basis in $\mathcal{H}_j$ is the eigenbasis of $\hat{J}_z = \frac{1}{2}(\hat{H}_1 - \hat{H}_2)$, with the usual quantum number m, so that if n_μ are the usual quantum numbers of the oscillators $\hat{H}_\mu$, then $n_1 = j + m, n_2 = j - m$. The simultaneous eigenstates of $\hat{\mathbf{J}}^2$ and $\hat{J}_z$ are $|jm\rangle$ or $|n_1 n_2\rangle$.

In the Nj-model we index the angular momenta with indices $r, s, \ldots = 1, \ldots, N$. The oscillators are now labelled $\hat{H}_{r\mu}$ with coordinates and momenta $\hat{x}_{r\mu}$ and $\hat{p}_{r\mu}$ and annihilation and creation operators $a_{r\mu}$ and $a^\dagger_{r\mu}$. The wavefunctions are now $\psi(x_{11}, x_{12}, x_{21}, \ldots, x_{N2})$ and the Hilbert space is $L^2(\mathbb{R}^{2N})$. We define operators

$$\hat{I}_r = \frac{1}{2} \sum_\mu a^\dagger_{r\mu} a_{r\mu}, \tag{8}$$

$$\hat{J}_{ri} = \frac{1}{2} \sum_{\mu\nu} a^\dagger_{r\mu} \sigma^i_{\mu\nu} a_{r\nu}, \tag{9}$$

$$\hat{\mathbf{J}}^2_r = \sum_i \hat{J}^2_{ri}, \tag{10}$$

$$\hat{J}_i = \sum_r \hat{J}_{ri} \qquad \text{or} \qquad \hat{\mathbf{J}} = \sum_r \hat{\mathbf{J}}_r, \tag{11}$$

$$\hat{\mathbf{J}}^2 = \sum_i \hat{J}^2_i, \tag{12}$$

most of which are obvious generalizations from the case $N = 1$. These satisfy the identity

$$\hat{\mathbf{J}}^2_r = \hat{I}_r(\hat{I}_r + 1), \tag{13}$$

and the commutation relations $[\hat{I}_r, \hat{J}_{si}] = \left[\hat{I}_r, \hat{\mathbf{J}}^2_s\right] = 0$. Each angular momentum vector $\hat{\mathbf{J}}_r$ also obeys the standard commutation relations among its components and square, which we omit, as does any sum of these angular momenta (partial or total).

The angular momenta generate an action of $[SU(2)]^N$ on the Hilbert space (one copy for each pair of oscillators). Here we discuss only the simultaneous rotation of all oscillator degrees of freedom by the same element of $SU(2)$, which is generated by the total angular momentum, but partial rotation operators can also be defined and are useful. We begin with the commutation relations,

$$[\hat{J}_i, a_{r\mu}] = -\frac{1}{2} \sum_\nu \sigma^i_{\mu\nu} a_{r\nu}, \tag{14}$$

$$\left[\hat{J}_i, a^\dagger_{r\mu}\right] = +\frac{1}{2} \sum_\nu a^\dagger_{r\nu} \sigma^i_{\nu\mu}, \tag{15}$$

which define the transformation properties of the operators $a_{r\mu}, a^\dagger_{r\mu}$ under infinitesimal rotations. We define a finite rotation operator in axis-angle or Euler angle form by

$$U(\mathbf{n}, \theta) = \exp(-\mathrm{i}\theta\, \mathbf{n} \cdot \mathbf{J}), \tag{16}$$

$$U(\alpha, \beta, \gamma) = U(\mathbf{z}, \alpha) U(\mathbf{y}, \beta) U(\mathbf{z}, \gamma), \tag{17}$$

where $\mathbf{n}$ is a unit vector defining an axis and θ an angle of rotation about that axis, and where $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are respectively the unit vectors along the three coordinate axes. The U operators form a faithful representation of $SU(2)$.

We use the symbol $u(\mathbf{n}, \theta)$ or $u(\alpha, \beta, \gamma)$ for the 2×2 matrices belonging to $SU(2)$, in axis-angle or Euler angle parameterization (not to be confused with the U operators that act on the Hilbert space of the $2N$ oscillators). Thus,

$$u(\mathbf{n}, \theta) = \exp(-\mathrm{i}\theta\, \mathbf{n} \cdot \boldsymbol{\sigma}/2) = \cos\theta/2 - \mathrm{i}\mathbf{n} \cdot \boldsymbol{\sigma} \sin\theta/2. \tag{18}$$

The exponentiated versions of equations (14) and (15) are

$$U^\dagger a_{r\mu} U = \sum_\nu u_{\mu\nu} a_{r\nu}, \qquad U^\dagger a^\dagger_{r\mu} U = \sum_\nu a^\dagger_{r\nu} (u^{-1})_{\nu\mu}, \tag{19}$$

where both U and u have the same parameterization. In the language of irreducible tensor operators the pair of operators (a_{r1}, a_{r2}) transforms as a spin-1/2 operator.

Similarly, vector operators are the angular momenta themselves, which satisfy the conjugation relations,

$$U^\dagger \hat{J}_{ri} U = \sum_j R_{ij} \hat{J}_{rj}, \tag{20}$$

where R is the 3×3 orthogonal rotation matrix with the same axis and angle as U. The relation between R and u (with the same axis and angle) is

$$R_{ij} = \tfrac{1}{2}\,\mathrm{tr}(U^\dagger \sigma_i U \sigma_j). \tag{21}$$

This is the usual projection from $SU(2)$ to $SO(3)$, in which the inverse image of a given $R \in SO(3)$ is a pair $(u, -u)$ in $SU(2)$.

4. The Wigner 3*j*-symbols in the Schwinger model

We now define the $3j$-symbols in the context of the Schwinger model. We take the $3j$-model, $N = 3$. One complete set of commuting observables on the Hilbert space $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ is $(\hat{I}_1, \hat{I}_2, \hat{I}_3, \hat{J}_{1z}, \hat{J}_{2z}, \hat{J}_{3z})$, with corresponding eigenstates $|j_1 j_2 j_3 m_1 m_2 m_3\rangle = |j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle$. Another complete set arises in the usual problem of addition of three angular momenta, in which we consider the values of j and m (the quantum numbers of $\hat{\mathbf{J}}^2$ and $\hat{J}_z$) that occur in the product space $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}$ for fixed values of (j_1, j_2, j_3), a subspace of $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$. The set of the five commuting operators $(\hat{I}_1, \hat{I}_2, \hat{I}_3, \hat{J}_3, \hat{\mathbf{J}}^2)$ that arises in this way is however not complete (the simultaneous eigenstates in general possess degeneracies), so to resolve these we introduce a sixth commuting operator, conventionally taken to be $\hat{\mathbf{J}}_{12}^2 = (\hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2)^2$ with quantum number j_{12} ($\hat{\mathbf{J}}_{23}^2$ or $\hat{\mathbf{J}}_{13}^2$ will also work).

The Wigner $3j$-symbols only involve the case $j = 0$, but we mention the others anyway because the foliation of the classical phase space into Lagrangian manifolds involves the other values. The usual rules for the addition of angular momenta show that if (j_1, j_2, j_3) satisfy the triangle inequality, then there exists precisely a one-dimensional subspace of $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}$ with $j = 0$; if they do not, then no such subspace exists. If we enlarge our point of view to the full Hilbert space $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, then there is an infinite-dimensional subspace with $j = 0$, a basis in which is specified by all triplets (j_1, j_2, j_3) that satisfy the triangle inequalities. If $j = 0$, then the quantum number m is superfluous, since $m = 0$; the quantum number j_{12} is superfluous as well, since $j_{12} = j_3$.

We note that if $\langle\psi|\hat{\mathbf{J}}^2|\psi\rangle = 0$ for any state $|\psi\rangle$, then $\hat{J}_i|\psi\rangle = 0$ for $i = 1, 2, 3$. Although the components of $\hat{\mathbf{J}}$ do not commute and so do not possess simultaneous eigenstates in general, the case of a state with $j = 0$ is an exception, since it is a simultaneous eigenstate of all three components of $\hat{\mathbf{J}}$ with eigenvalues 0. With this in mind we denote the basis of states in the subspace of the full Hilbert space with $j = 0$ by $|j_1 j_2 j_3 \mathbf{0}\rangle$, where the zero vector $\mathbf{0}$ indicates the vanishing eigenvalues of $\hat{\mathbf{J}}$. These basis states are also eigenstates of the operators $\hat{\mathbf{J}}_{ij}^2$, for example, $\hat{\mathbf{J}}_{12}^2 |j_1 j_2 j_3 \mathbf{0}\rangle = j_3(j_3 + 1)|j_1 j_2 j_3 \mathbf{0}\rangle$.

When the phase of the state $|j_1 j_2 j_3 \mathbf{0}\rangle$ is chosen to agree with Wigner's convention for the phases of the 3j-symbols, we have

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \langle j_1 j_2 j_3 m_1 m_2 m_3 | j_1 j_2 j_3 \mathbf{0}\rangle. \tag{22}$$

In this manner we have expressed the 3j-symbol as a matrix element connecting the eigenstates of two sets of observables, $(\hat{I}_1, \hat{I}_2, \hat{I}_3, \hat{J}_{1z}, \hat{J}_{2z}, \hat{J}_{3z})$ on the left and $(\hat{I}_1, \hat{I}_2, \hat{I}_3, \hat{J}_x, \hat{J}_y, \hat{J}_z)$ on the right. Since the second set is noncommuting, we will require a generalization of (3) to compute the semiclassical approximation to the 3j-symbols.

5. Classical mechanics of the Schwinger model

The classical mechanics of the Schwinger model must be well understood in order to carry out a semiclassical analysis. A general reference on the classical mechanics of integrable systems from the modern point of view is Cushman and Bates (1997), where harmonic oscillators in particular are treated.

5.1. The 1j-model

We start with the 1j-model, defining two classical oscillators $H_\mu = (1/2)\left(x_\mu^2 + p_\mu^2\right)$ and $H = \sum_\mu H_\mu$, as in the quantum case. The classical configuration space is $\mathbb{R}^2$ and the phase space is $\mathbb{R}^4$. We introduce complex coordinates on phase space $z_\mu = (x_\mu + \mathrm{i}p_\mu)/\sqrt{2}$ and $\bar{z}_\mu = (x_\mu - \mathrm{i}p_\mu)/\sqrt{2}$, where we use an overbar for complex conjugation. These are the Weyl symbols of the operators $a_\mu, a_\mu^\dagger$. The complex coordinates $z_\mu, \bar{z}_\mu$ allow us to identify the phase space $\mathbb{R}^4$ with $\mathbb{C}^2$, that is, knowledge of z_1 and z_2 allows us to find all four real coordinates (x_1, x_2, p_1, p_2), since $\bar{z}$'s are complex conjugates of z's. As we shall see, coordinates (z_1, z_2), arranged as a two-component column vector, transform as a spinor under certain $SU(2)$ transformations. Variables z_μ and $\mathrm{i}\bar{z}_\mu$ are canonically conjugate (q's and p's, respectively), so that the Poisson bracket of two functions f and g on phase space can be written as

$$\begin{aligned}\{f, g\} &= \sum_\mu \left(\frac{\partial f}{\partial x_\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial x_\mu}\right) \\ &= \sum_\mu \left(\frac{\partial f}{\partial z_\mu}\frac{\partial g}{\partial(\mathrm{i}\bar{z}_\mu)} - \frac{\partial f}{\partial(\mathrm{i}\bar{z}_\mu)}\frac{\partial g}{\partial z_\mu}\right).\end{aligned} \tag{23}$$

The basic building blocks of the classical Schwinger model are the function

$$I = \frac{1}{2}\sum_\mu \bar{z}_\mu z_\mu = \frac{1}{2}\sum_\mu |z_\mu|^2, \tag{24}$$

and the three functions

$$J_i = \frac{1}{2}\sum_{\mu\nu} \bar{z}_\mu \sigma^i_{\mu\nu} z_\nu, \tag{25}$$

for $i = 1, 2, 3$, which define a classical angular momentum vector. We also define $\mathbf{J}^2 = \sum_i J_i^2$. These functions satisfy the identity $\mathbf{J}^2 = I^2$ and the Poisson bracket relations $\{I, J_i\} = 0$, $\{J_i, J_j\} = \sum_k \epsilon_{ijk} J_k$, $\{J_i, \mathbf{J}^2\} = 0$.

There are two groups of interest that act on the phase space $\mathbb{R}^4$ or $\mathbb{C}^2$. The first is $U(1)$, generated by I. Hamilton's equations for I are

$$\frac{\mathrm{d}z_\mu}{\mathrm{d}\psi} = \frac{\partial I}{\partial(\mathrm{i}\bar{z}_\mu)} = -\frac{\mathrm{i}}{2}z_\mu, \qquad \frac{\mathrm{d}(\mathrm{i}\bar{z}_\mu)}{\mathrm{d}\psi} = -\frac{\partial I}{\partial z_\mu} = -\frac{1}{2}\bar{z}_\mu, \tag{26}$$

where ψ is the parameter of the orbits. These have the solution

$$z_\mu(\psi) = \exp(-\mathrm{i}\psi/2) z_\mu(0), \qquad \bar{z}_\mu(\psi) = \exp(\mathrm{i}\psi/2)\bar{z}_\mu(0). \tag{27}$$

Under the I-flow, the two-component spinor (z_1, z_2) just gets multiplied by an overall phase $\exp(-\mathrm{i}\psi/2)$. Except for the special initial condition $(z_1, z_2) = (0, 0)$ (the origin of phase space $\mathbb{R}^4$ or $\mathbb{C}^2$), the orbits are circles with period 4π with respect to the variable ψ. Henceforth when citing equations such as (26) or (27) we shall omit the second half, when it is simply the complex conjugate of the first half.

We denote a value of I by $j \geqslant 0$. This is convenient notation, but in this classical context j is a continuous variable not to be identified with the quantum number of any operator (see section 10). Except for the origin $j = 0$, the level set $I = j$ (or equivalently, $\mathbf{J}^2 = j^2$) is the sphere S^3, which is foliated into circles by the action (27). This foliation is precisely the Hopf fibration (Frankel 1997, Nakahara 2003), yielding the quotient space $S^2 = S^3/S^1$.

The second group acting on phase space is $SU(2)$, whose action is generated by J_i. Explicitly, if $\mathbf{n}$ is a unit vector and θ an angle, then the solutions of Hamilton's equations

$$\frac{\mathrm{d}z_\mu}{\mathrm{d}\theta} = \frac{\partial(\mathbf{n}\cdot\mathbf{J})}{\partial(\mathrm{i}\bar{z}_\mu)} = -\frac{\mathrm{i}}{2}\sum_\nu (\mathbf{n}\cdot\boldsymbol{\sigma})_{\mu\nu} z_\nu \tag{28}$$

and its complex conjugate are

$$z_\mu(\theta) = \sum_\nu u(\mathbf{n}, \theta)_{\mu\nu} z_\nu(0) \tag{29}$$

and its complex conjugate. These are the obvious classical analogues of equations (19); note that the period in θ is 4π. It is because of this $SU(2)$ action that we say that coordinates (z_1, z_2) form a spinor. This classical action of $SU(2)$ can be understood as a subgroup of the classical group of linear canonical transformations, $Sp(4)$ (Littlejohn 1986); in general, $Sp(2N)$ possesses a subgroup $Sp(2N) \cap O(2N)$ that is isomorphic to $U(N)$, which contains the subgroup $SU(N)$ (in this case, $N = 2$). When the symplectic matrices lying in the $SU(2)$ subgroup are expressed in the complex basis $(z_\mu, \mathrm{i}\bar{z}_\mu)$, they block diagonalize with u multiplying z's and $\bar{u}$ multiplying $\bar{z}$'s.

Equation (25) defines a map (a projection) $\pi : \mathbb{R}^4$ (or $\mathbb{C}^2$) $\to \mathbb{R}^3$, where $\mathbb{R}^3$ is 'angular momentum space', the space with coordinates (J_1, J_2, J_3). Here and below we use π to denote this map or its generalization to the Nj-model. The map π maps a larger space onto a smaller one, and so is not one-to-one. The inverse image of a point $\mathbf{J}$ of angular momentum space is a set of spinors that differ by an overall phase. It is easy to see that the definition (25) does not depend on the overall phase of the spinor. Thus, the inverse image is a circle, except in the case $\mathbf{J} = 0$ when it is a single point (the origin of phase space $\mathbb{C}^2$ or $\mathbb{R}^4$).

These circles are precisely the orbits of the I-flow (27). Any function f that is constant on these circles projects onto a well-defined function on angular momentum space. But such functions are those that Poisson commute with I, $\{f, I\} = 0$. This includes I itself as well as the three J_i. We can write such a function as $f(z_1, z_2, \bar{z}_1, \bar{z}_2)$ or $f(\mathbf{J})$. Now if f and g are any two such functions, then so is their Poisson bracket $\{f, g\}$, as follows from the Jacobi identity, $\{\{f, g\}, I\} = \{f, \{g, I\}\} + \{g, \{I, f\}\} = 0$. Thus, this Poisson bracket can be computed directly in angular momentum space without going back to the bracket (23); the result is the Lie–Poisson bracket,

$$\{f, g\} = \mathbf{J}\cdot\left(\frac{\partial f}{\partial \mathbf{J}} \times \frac{\partial g}{\partial \mathbf{J}}\right). \tag{30}$$

Interpretations of these spaces may be given in terms of the theory of 'reduction' (Marsden and Ratiu 1999). Angular momentum space is the Poisson manifold that results from Poisson reduction of the phase space $\mathbb{R}^4$ under the $U(1)$ action (27) generated by I. It is not by itself an ordinary phase space (symplectic manifold), which would have an even dimensionality, but it is foliated into symplectic submanifolds (the symplectic leaves). In this case, the symplectic leaves are the 2-spheres in angular momentum space, that is, the level sets $\mathbf{J}^2 = j^2$, the images under π of the 3-spheres $I = j$ in $\mathbb{R}^4$ or $\mathbb{C}^2$. Canonical coordinates on a given 2-sphere $\mathbf{J}^2 = j^2$ are (ϕ, J_z), a (q, p) pair, where $J_z = j\cos\theta$ and where (θ, ϕ) are the usual spherical angles in angular momentum space. Thus we have

$$\mathrm{d}q \wedge \mathrm{d}p = \mathrm{d}\phi \wedge \mathrm{d}(j\cos\theta) = j\sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi = j\, \mathrm{d}\Omega, \tag{31}$$

and the symplectic form on a given sphere is $j\,\mathrm{d}\Omega$, where $\mathrm{d}\Omega$ is the element of solid angle. This is not the geometrical solid angle in a Euclidean geometry on angular momentum space, which would be $j^2\mathrm{d}\Omega$. Another interpretation of angular momentum space is that it is the dual of the Lie algebra of $SU(2)$, while π, given by (25), is the momentum map of the $SU(2)$ action (29).

We now have three spaces, the 'large phase space' $\mathbb{R}^4$ or $\mathbb{C}^2$, its image under π, 'angular momentum space' $\mathbb{R}^3$ and its symplectic leaves, the 'small phase spaces', the 2-spheres $\mathbf{J}^2 = j^2$. Angular momentum space is useful for visualizing classical angular momentum vectors, but by considering inverse projections under π the corresponding geometrical objects in the large phase space can be constructed. Angular momentum space has been used since the time of the old quantum theory for visualizing the classical limit of quantum angular momentum operators; for example, one spoke of an angular momentum vector 'precessing' around the z-direction. In reality, the 'precession' defines a manifold of classical states in the small phase space that is a level set of a complete set of commuting observables, that is, it is an invariant torus of an integrable system (just a circle in the 1j-model, where the commuting observables are I and J_z).

5.2. The Nj-model

We now consider the classical mechanics of the Nj-model, which is mostly a simple generalization of the 1j-model. We have $2N$ classical oscillators $H_{r\mu} = (1/2)\big(x_{r\mu}^2 + p_{r\mu}^2\big)$; the configuration space is $(\mathbb{R}^2)^N = \mathbb{R}^{2N}$ and the 'large' phase space is $(\mathbb{R}^4)^N = \mathbb{R}^{4N}$ or $(\mathbb{C}^2)^N = \mathbb{C}^{2N}$. We define $z_{r\mu} = (x_{r\mu} + \mathrm{i}p_{r\mu})/\sqrt{2}$, $\bar{z}_{r\mu} = (x_{r\mu} - \mathrm{i}p_{r\mu})/\sqrt{2}$, so a point in phase space can be thought of as a collection of N 2-spinors, (z_{r1}, z_{r2}), $r = 1, \ldots, N$. We make the obvious definitions (classical versions of equations (8)–(12)),

$$I_r = \frac{1}{2}\sum_{\mu} |z_{r\mu}|^2, \tag{32}$$

$$J_{ri} = \frac{1}{2}\sum_{\mu\nu} \bar{z}_{r\mu}\sigma^i_{\mu\nu} z_{r\nu}, \tag{33}$$

as well as $\mathbf{J}_r^2 = \sum_i J_{ri}^2$, $J_i = \sum_r J_{ri}$ or $\mathbf{J} = \sum_r \mathbf{J}_r$ and $\mathbf{J}^2 = \sum_i J_i^2$.

We denote a value of the functions I_r by $j_r \geqslant 0$; for positive values $j_r > 0$, $r = 1, \ldots, N$, the level set $I_r = j_r$ (or $\mathbf{J}_r^2 = j_r^2$) in the phase space $(\mathbb{R}^4)^N$ is $S^3 \times \cdots \times S^3 = (S^3)^N$. The flow generated by I_r for a specific value of r is just multiplication of the rth spinor (z_{r1}, z_{r2}) by a phase factor $\exp(-\mathrm{i}\psi_r/2)$, as in (27); the other spinors are not affected. Thus, the N commuting flows generated by all I_r's constitute a $U(1)^N = T^N$ action on the large phase space (T^N is the N-torus).

Equation (33) defines the projection map $\pi : (\mathbb{C}^2)^N \to (\mathbb{R}^3)^N$, the latter space being 'angular momentum space' for the Nj-model, with one copy of $\mathbb{R}^3$ for each classical angular momentum vector $\mathbf{J}_r$. In view of its importance, we write out the components of this map explicitly:

$$J_{rx} = \frac{1}{2}(\bar{z}_{r1}z_{r2} + \bar{z}_{r2}z_{r1}) = \mathrm{Re}(\bar{z}_{r1}z_{r2}), \tag{34}$$

$$J_{ry} = -\frac{\mathrm{i}}{2}(\bar{z}_{r1}z_{r2} - \bar{z}_{r2}z_{r1}) = \mathrm{Im}(\bar{z}_{r1}z_{r2}), \tag{35}$$

$$J_{rz} = \frac{1}{2}(|z_{r1}|^2 - |z_{r2}|^2). \tag{36}$$

Points of angular momentum space can be visualized as N classical angular momentum vectors, each living in its own angular momentum space or N such vectors all in the same three-dimensional angular momentum space. The inverse image under π of a set of N nonvanishing classical angular momentum vectors is an N-torus in the large phase space, generated by taking any point in the inverse image (a collection of N 2-spinors), and multiplying them by N independent, overall phase factors. We denote the angles on this torus by ψ_r, $r = 1, \ldots, N$, which are the evolution parameters corresponding to I_r, as in (27); thus their periods are 4π. As in the $1j$-model, angular momentum space $(\mathbb{R}^3)^N$ is a Poisson manifold, now with Poisson bracket

$$\{f, g\} = \sum_r \mathbf{J}_r \cdot \left(\frac{\partial f}{\partial \mathbf{J}_r} \times \frac{\partial g}{\partial \mathbf{J}_r}\right). \tag{37}$$

The symplectic leaves (the 'small phase spaces') are the spaces $S^2 \times \cdots \times S^2 = (S^2)^N$ obtained by fixing the values of $j_1, \ldots, j_N$, with canonical coordinates (ϕ_r, J_{rz}) on each sphere.

In the classical Nj-model any partial or total sum of the angular momenta $\mathbf{J}_r$ generates an $SU(2)$ action on the large phase space, generalizing equations (28) and (29) in the $1j$-model, in that the $SU(2)$ matrix u is applied to all spinors (z_{r1}, z_{r2}) whose r values lie in the sum. For example, the total $\mathbf{J}$ rotates all spinors.

These $SU(2)$ actions on the large phase space project to $SO(3)$ actions on angular momentum space. Consider, for example, the $SU(2)$ action generated by the total $\mathbf{J}$. Along an orbit in the large phase space generated by $\mathbf{n} \cdot \mathbf{J}$, parameterized by θ, we can follow the value of $\mathbf{J}_r$, giving us $\mathbf{J}_r(\theta)$, an orbit in the small phase space (the projection under π of the first orbit). The latter orbit is

$$J_{ri}(\theta) = \sum_j R(\mathbf{n}, \theta)_{ij} J_{rj}(0), \tag{38}$$

where $R(\mathbf{n}, \theta)$ is the 3×3 rotation associated with $u(\mathbf{n}, \theta)$ according to (21). This is the classical analogue of (20). It follows from (33) and the spinor adjoint equation, $u^\dagger \sigma^i u = \sum_j R_{ij} \sigma^j$, itself equivalent to (21). Thus, under the $SU(2)$ action on the large phase space generated by $\mathbf{J}$, the individual vectors $\mathbf{J}_r$ rotate in their individual angular momentum spaces by the corresponding 3×3 rotation. For example, J_z rotates all vectors $\mathbf{J}_r$ about the z-axis. Because of the two-to-one relation between $SU(2)$ and $SO(3)$, when the orbit in the large phase space goes around once (θ goes from 0 to 4π), the angular momentum vectors go around twice in their individual angular momentum spaces.

We may visualize this action as in figure 1, where A represents a point of angular momentum space (a set of N classical angular momentum vectors $\mathbf{J}_r$ in the Nj-model). To obtain the generic case we assume these vectors are linearly independent (in particular, none of them vanishes). In the figure, T is the inverse image of A under π, an N-torus. Point a is any specific point in the large phase space on this N-torus, to which the $SU(2)$ rotation $u(\mathbf{n}, \theta)$ is applied for $0 \leqslant \theta < 4\pi$. That is, we treat a as initial conditions for the Hamiltonian flow

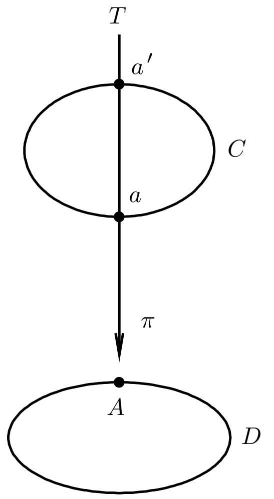

Figure 1. The action of an $SU(2)$ rotation about a fixed axis on a point of the large phase space and its projection onto angular momentum space.

generated by $\mathbf{n} \cdot \mathbf{J}$, with θ as the parameter. This generates the circle C in the large phase space, which amounts to rotating all N spinors by the same $u(\mathbf{n}, \theta)$. The projection of the circle C is a circle D in angular momentum space $(\mathbb{R}^3)^N$, that is, all classical vectors $\mathbf{J}_r$ rotate about $\mathbf{n}$ by angle θ. However, when the circle C is covered once, circle D is covered twice. This is because when $\theta = 2\pi$, the spinor rotation $u(\mathbf{n}, \theta) = -1$, so all spinors in the large phase space are just multiplied by -1. This is illustrated as point a' in the figure, where all spinors are -1 times their values at a. Since -1 is just a phase factor, both a and a' project onto the same point A in angular momentum space. These are the only two points on C that project onto A; for θ not a multiple of 2π, the spinor rotation $u(\mathbf{n}, \theta)$ is not a multiplication by a phase factor.

Alternatively, we may apply the entire group $SU(2)$ to the original point a (not just rotations along a fixed axis). Then the manifold C is the orbit of the $SU(2)$ action which is diffeomorphic to $SU(2)$. The point a' is the image of a under $u = -1$, a specific element in $SU(2)$, and once again it projects onto the original point A in the small phase space. The manifold D is the orbit of point A under the group $SO(3)$. In the $1j$-model, it is just a sphere in angular momentum space (all vectors that can be reached from the original one by applying all rotations), while in the Nj-model for $N > 1D$ is generically diffeomorphic to $SO(3)$ (it is the set of all classical configurations of N angular momentum vectors that can be obtained from the original one by applying rigid rotations).

6. The invariant jm-tori

In this section we continue with the classical point of view, examining the classical manifolds corresponding to the left-hand side of the matrix element (22). For this exercise and the rest of the paper we adopt a $3j$-model ($N = 3$). The manifolds in question are the level sets of the commuting functions I_r, J_{rz}, $r = 1, 2, 3$, or, equally well, of the functions $I_{r\mu} = (1/2)|z_{r\mu}|^2$ for $r = 1, 2, 3$, $\mu = 1, 2$, since $I_r = I_{r1} + I_{r2}$ and $J_{rz} = I_{r1} - I_{r2}$. We denote the level sets by $I_r = j_r$, $J_{rz} = m_r$ for contour values j_r, m_r, $r = 1, 2, 3$, or, equivalently, by

$$I_{r1} = \tfrac{1}{2}(j_r + m_r), \qquad I_{r2} = \tfrac{1}{2}(j_r - m_r). \tag{39}$$

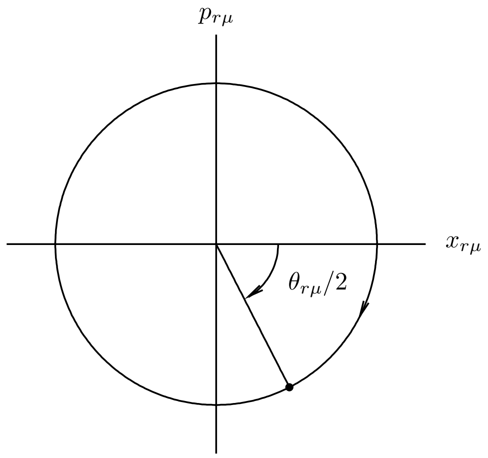

Figure 2. Harmonic oscillator motion in the $x_{r\mu}$–$p_{r\mu}$ plane, generated by $I_{r\mu}$. The definition of angle $\theta_{r\mu}$ is shown.

In spite of the notation, j_r and m_r take on continuous values and are not necessarily the eigenvalues of any quantum operators. Since $I_{r\mu}$ are all nonnegative, we have

$$j_r \geqslant 0, \qquad -j_r \leqslant m_r \leqslant j_r, \tag{40}$$

the classical analogues of the usual inequalities in quantum mechanics.

Since each of the six $I_{r\mu}$ is a harmonic oscillator (times 1/2), the level set of $I_{r\mu}$'s is an invariant torus of a collection of harmonic oscillators. Generically (for nonzero amplitude in each oscillator, that is, when none of the quantities $j_r \pm m_r$ vanishes) this is a 6-torus, upon which the coordinates may be taken to be the six angles $\theta_{r\mu}$, the variables of evolution of $I_{r\mu}$. The Hamiltonian flow generated by $I_{r\mu}$ for a specific value of r and μ just multiplies $z_{r\mu}$ for the same values of r and μ by $\exp(-\mathrm{i}\theta_{r\mu}/2)$, while leaving all other z's unaffected. This is not an overall spinor rotation since the other half of the spinor containing the given $z_{r\mu}$ is not affected. If viewed in the Cartesian $x_{r\mu}$–$p_{r\mu}$ phase plane, this flow is a clockwise rotation by angle $\theta_{r\mu}/2$, as illustrated in figure 2. The period of the angles $\theta_{r\mu}$ is 4π. We agree to measure the angles $\theta_{r\mu}$ from the positive $x_{r\mu}$ axis, as in the figure, where $z_{r\mu}$ is real and positive (or zero); this is a specific convention for a set of canonical coordinates $(\theta_{r\mu}, I_{r\mu})$, $r = 1, 2, 3$, $\mu = 1, 2$, on the large phase space. The volume of the 6-torus with respect to the measure $\mathrm{d}\theta_{11} \wedge \cdots \wedge \mathrm{d}\theta_{32}$ is $(4\pi)^6$.

These tori are also the orbits of the flows generated by the observables I_r, J_{rz}. We denote the evolution variables of I_r and J_{rz} by ψ_r (as above) and ϕ_r, respectively. Each J_{rz} generates an $SU(2)$ rotation about the z-axis on the spinor with the given value of r; thus each of the six angles (ψ_r, ϕ_r) has period 4π. However, when we allow all six angles (ψ_r, ϕ_r) to range from 0 to 4π, the torus is actually covered eight times. This can be seen from figure 1: a rotation by 2π in one of ϕ's and one of ψ's returns us to the initial point (the path is a to a' along C in figure 1, then a' to a along T.) Alternatively, we may consider the canonical transformation $(\theta_{r\mu}; I_{r\mu}) \to (\psi_r, \phi_r; I_r, J_{rz})$, generated by

$$F_2(\theta_{r1}, \theta_{r2}, I_r, J_{rz}) = \tfrac{1}{2}[\theta_{r1}(I_r + J_{rz}) + \theta_{r2}(I_r - J_{rz})], \tag{41}$$

which generates $I_r = I_{r1} + I_{r2}$, $J_{rz} = I_{r1} - I_{r2}$ and

$$\psi_r = \tfrac{1}{2}(\theta_{r1} + \theta_{r2}), \qquad \phi_r = \tfrac{1}{2}(\theta_{r1} - \theta_{r2}), \tag{42}$$

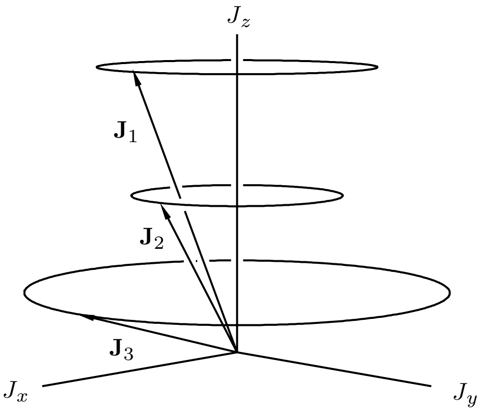

Figure 3. The quantum state $|j_1 j_2 j_3 m_1 m_2 m_3\rangle$ corresponds classically to a 3-torus in the small phase space $S^2 \times S^2 \times S^2$, which can be visualized as three $\mathbf{J}$ vectors lying on three cones with fixed values of J_z. The three azimuthal angles are independent.

so that the Jacobian in the angles is $(1/2)^3 = 1/8$. To cover the torus precisely once we may let ψ_r's range from 0 to 4π and ϕ_r's from 0 to 2π or vice versa; thus the volume of the torus with respect to $\mathrm{d}\psi_1 \wedge \mathrm{d}\psi_2 \wedge \mathrm{d}\psi_3 \wedge \mathrm{d}\phi_1 \wedge \mathrm{d}\phi_2 \wedge \mathrm{d}\phi_3$ is

$$V_{jm} = (2\pi)^3(4\pi)^3. \tag{43}$$

The angles ϕ_r defined in this way on the large phase space can be projected onto the small phase space, whereupon they coincide with the usual azimuthal spherical angles in the individual angular momentum spaces. It is clear this must be so to within an additive constant, since the J_{rz}-flow is just an $SO(3)$ rotation about the z-axis in the rth angular momentum space, but by our conventions even the additive constant comes out right. To see this we note first of all that ϕ_r for a given r is constant along the I_s-flows for all s, since the variables in question are members of a canonical coordinate system on the large phase space and satisfy $\{\phi_r, I_s\} = 0$. Thus, ϕ_r, defined in the large phase space, projects onto a meaningful function in angular momentum space. Next, to compute the value of ϕ_r for a specific angular momentum vector $\mathbf{J}_r$, it suffices to take any point in the 3-torus that is the inverse image, that is, any value of the angles ψ_s may be chosen. For simplicity we take $\psi_r = 0$, which implies $\theta_{r1} = \phi_r$ and $\theta_{r2} = -\phi_r$. This in turn implies $z_{r1} = |z_{r1}|\exp(-\mathrm{i}\phi_r/2)$, $z_{r2} = |z_{r2}|\exp(\mathrm{i}\phi_r/2)$. But by equations (34) and (35), these imply $J_{rx} = J_{r\perp}\cos\phi_r$, $J_{ry} = J_{r\perp}\sin\phi_r$, where $J_{r\perp} = |z_{r1}||z_{r2}|$.

We shall henceforth call the level set $I_r = j_r$, $J_{rz} = m_r$ the 'jm-torus'. This torus can be projected onto angular momentum space; we consider the generic case when $j_r \pm m_r \neq 0$ for all r, in which case the jm-torus is a 6-torus. In this case, its image in angular momentum space is a 3-torus, which, since it is a surface on which $I_r = j_r$, is also a submanifold of the small phase space. This is because the three ψ_r angles just change the overall phases of the three spinors, without changing their image under π, so the three coordinates on the projected 3-torus are the angles ϕ_r. The 3-torus in angular momentum space can be visualized as three classical vectors $\mathbf{J}_r$ in a single angular momentum space, with specified values of $m_r = J_{rz}$, 'precessing' about the z-axis, see figure 3. This is an example of how we shall visualize manifolds in the large phase space: the jm-torus, a six-dimensional manifold in the large phase space (itself with 12 dimensions), is visualized as three angular momentum vectors in three-dimensional space, as in figure 3, defining a 3-torus by varying their azimuthal angles independently, and each point of this 3-torus is associated with another 3-torus, the inverse projection under π of the given point, which consists of independently changing the overall phases of the three spinors. The six-dimensional jm-torus is thus conceived of as the Cartesian product $T^3 \times T^3$ (it is actually a trivial bundle).

7. The Wigner manifold

Now we turn to the right-hand side of the matrix element (22), containing the state $|j_1 j_2 j_3 \mathbf{0}\rangle$. This state suggests that we examine the classical manifold upon which I_r have definite values, say, $I_r = j_r$, and upon which $\mathbf{J}^2 = 0$. Again, we do not necessarily identify j_r with any quantum numbers, but it is convenient in the following to assume that none of j_r's vanishes.

7.1. Properties of the Wigner manifold

Usually, the dimensionality of a manifold can be guessed by counting the constraints that define it, for example, we expect the manifold in the large (12-dimensional) phase space upon which $I_r = j_r, \mathbf{J}_{12}^2 = j_{12}^2, J_z = m$ and $\mathbf{J}^2 = j^2$, for given contour values, to be six-dimensional (six constraints on 12 variables). Indeed, for most values of j this is correct, and the manifold in question is a 6-torus (by the Liouville–Arnold theorem, for certain ranges of the contour values). These are the invariant tori that would be involved in the semiclassical treatment of the addition of three angular momenta, producing a nonzero result (the case $j \neq 0$). But this naive dimension count only works when the differentials of the functions in question are linearly independent (in particular, nonvanishing) on the manifold. This condition breaks down when $\mathbf{J}^2 = j^2 = 0$, since $\mathbf{J}^2 = 0$ and $\mathbf{J} = 0$ imply one another, and $d(\mathbf{J}^2) = 2\mathbf{J} \cdot d\mathbf{J} = 0$. In fact, just the four conditions $I_r = j_r > 0, \mathbf{J}^2 = j^2 = 0$ define a six-dimensional manifold in the 12-dimensional large phase space (for certain ranges of j_r). To see this we note first that since $\mathbf{J}^2 = 0$ implies $J_i = 0, i = 1, 2, 3$, and $J_z = m = 0$ in particular, the J_z constraint is not independent; neither is the $\mathbf{J}_{12}^2 = j_{12}^2$ constraint, since when $\mathbf{J} = 0$, $j_{12} = j_3$. This is just as in the quantum case. In fact, the manifold $I_r = j_r, \mathbf{J}^2 = 0$ is characterized equivalently but better by $I_r = j_r, \mathbf{J} = 0$, since the six differentials $\mathrm{d}I_r$ and $\mathrm{d}J_i$ are linearly independent of it. (Although J_i vanish on the manifold in question, their differentials do not.) Thus, the naive count of dimensions works with the set I_r, J_i.

We shall call the manifold $I_r = j_r, \mathbf{J} = 0$ in the large phase space the 'Wigner manifold', because it corresponds to the rotationally invariant state $|j_1 j_2 j_3 \mathbf{0}\rangle$ introduced by Wigner in his definition of the $3j$-symbols. The dimensionality of this manifold (six, in the appropriate ranges of j_r's) is the same as that of the invariant tori of any integrable system of six degrees of freedom, and indeed the same as that of the nearby invariant tori in phase space corresponding to the level sets of the functions $(I_1, I_2, I_3, \mathbf{J}_{12}^2, J_z, \mathbf{J}^2)$ when $j > 0$. The Wigner manifold, however, is not a torus. This is not a contradiction of the Liouville–Arnold theorem, which requires that the classical observables making up the level set should Poisson commute. In the case of the Wigner manifold, we do have $\{I_r, I_s\} = 0$ and $\{I_r, J_i\} = 0$, but $\{J_i, J_j\} = \sum_k \epsilon_{ijk} J_k$. The Hamiltonian vector fields corresponding to the functions I_r, J_i are linearly independent of the Wigner manifold, but the three J_i-flows do not commute (two Hamiltonian flows commute if and only if their Poisson bracket is a constant). The Wigner manifold is, however, an orbit of the collective action of these Hamiltonian flows (any point can be reached from any other point by following the flows in some order). These facts, information about the topology of the Wigner manifold, and the required ranges on the contour values j_r will be clarified momentarily.

The Wigner manifold is also a Lagrangian manifold, like the invariant tori of an integrable system. This means that the integral of $p\,\mathrm{d}x$ along the manifold is locally independent of path, so an action function $S(x, A)$ can be defined. This function in turn is the solution of the

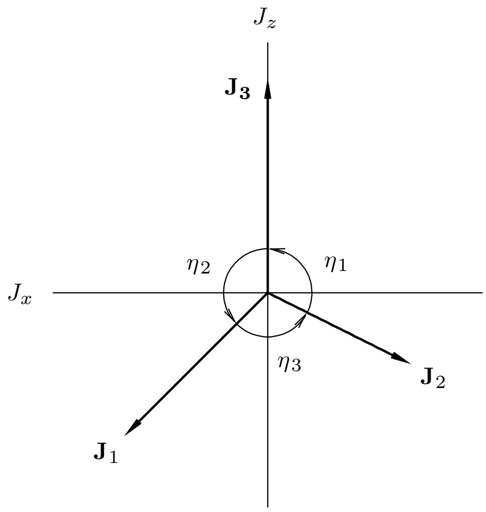

Figure 4. If j_1, j_2 and j_3 satisfy the triangle inequalities, then they define a triangle that is unique apart from its orientation. A standard orientation places the triangle in the x–z plane with sides $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$ oriented as shown. The angle opposite $\mathbf{J}_r$ is η_r.

simultaneous Hamilton–Jacobi equations for the observables $(I_1, I_2, I_3, J_x, J_y, J_z)$, call them A_i, $i = 1, \ldots, 6$ for short, of which the Wigner manifold is the level set.

To prove that the Wigner manifold is Lagrangian, we note that the differentials $\mathrm{d}A_i$ are linearly independent, so the vector fields X_i are too, and span the tangent space to the Wigner manifold at each point. Evaluating the symplectic form on these vector fields, we have $\omega(X_i, X_j) = -\{A_i, A_j\}$. These Poisson brackets all vanish except for $\{J_i, J_j\}$; the latter are nonzero at most points in phase space, but on the Wigner manifold where $\mathbf{J} = 0$ these also vanish. Thus the symplectic form restricted to the Wigner manifold vanishes, the condition that the Wigner manifold be Lagrangian.

To visualize the Wigner manifold we work our way up from angular momentum space to the large phase space. First we attempt to construct three angular momentum vectors of given positive lengths j_1, j_2, j_3 that add up to the zero vector. This can be done if and only if j_r satisfy the triangle inequalities, whereupon the values of j_r's (the lengths of the sides) specify a triangle that is unique to within orientation. If we choose a standard or reference orientation for the triangle, then the three desired vectors are the vectors running along its sides. Let us therefore assume that the triangle inequalities are satisfied, and let us choose a standard orientation for the triangle by placing the $\mathbf{J}_3$ along the z-axis, $\mathbf{J}_1$ in the x–z plane with $J_{1x} > 0$, and $\mathbf{J}_2$ in the x–z plane with $J_{2x} < 0$, as illustrated in figure 4. Given any two triangles with the same (positive) sides, there exists a unique rotation that maps one into the other; this fact and others regarding triangles are discussed in the context of the three-body problem by Littlejohn and Reinsch (1995). In the present context this means that all classical configurations of three classical angular momentum vectors of fixed lengths that add up to the zero vector are related to any one such configuration, such as the one shown in figure 4, by a unique rotation. Thus the manifold of such classical configurations in angular momentum space $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ or in the small phase space $S^2 \times S^2 \times S^2$ is diffeomorphic to $SO(3)$.

The Wigner manifold in the large phase space is now the inverse projection under π of this $SO(3)$ manifold in angular momentum space. Since the inverse image of any given point of angular momentum space is a 3-torus in the large phase space (obtained by varying the

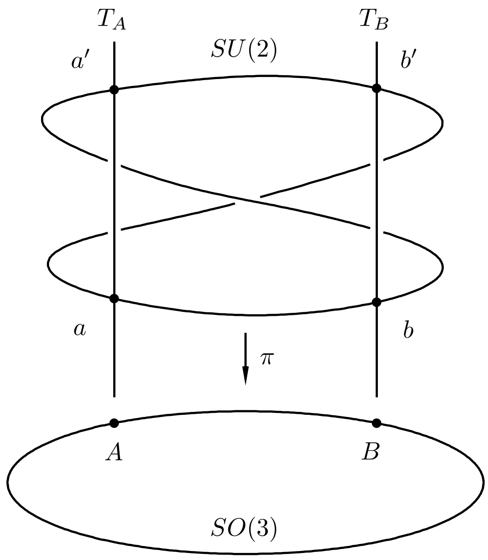

Figure 5. A schematic illustration showing how the Wigner manifold in the large phase space is the inverse image under π of a set of triangles formed from three angular momentum vectors with vanishing sum, all related by rigid rotations.

overall phases of the three spinors), the Wigner manifold is a 3-torus bundle over $SO(3)$ and is six-dimensional. The bundle is nontrivial.

The Wigner manifold may also be visualized with the help of figure 5, an elaboration of figure 1. It is assumed that the three j's are positive and satisfy the triangle inequality. The lower part of this figure refers to angular momentum space, while the upper part refers to the large phase space. Projection π maps between the two spaces. Point A in angular momentum space is a state of three classical angular momenta of the given lengths j_r whose vector sum is zero (that is, the angular momenta define a triangle), in a definite orientation. To be specific, let us say that A is the configuration shown in figure 4. By applying all $SO(3)$ rotations to A we generate all orientations of the triangle, of which B in the figure is one. The lower circle in the figure represents the manifold of such configurations, diffeomorphic to $SO(3)$.

The inverse image of any point on this manifold under π is a 3-torus in the large phase space. The 3-tori above points A and B are indicated schematically as lines T_A and T_B in the figure. Let a be some point on T_A. To be specific, if A is the configuration of angular momentum vectors shown in figure 4 and the stated conditions on j_r hold, then none of the three vectors lies on the negative z-axis. This means that for any point on T_A, $|z_{r1}|^2$ is never zero, since by equations (32) and (36) we have

$$|z_{r1}|^2 = j_r + J_{rz}, \qquad |z_{r2}|^2 = j_r - J_{rz}, \tag{44}$$

for $r = 1, 2, 3$. Thus by adjusting the overall phases of the three spinors (z_{r1}, z_{r2}), we can make z_{r1} real and positive for all $r = 1, 2, 3$. Let this be the point a in figure 5.

It is notationally tempting to write m_r for the value of J_{rz}, but we shall not do this in the context of the Wigner manifold, instead reserving the symbol m_r for the contour value of J_{rz} on the jm-torus.

Now we apply spinor rotations to point a, that is, simultaneous multiplication of all three spinors (z_{r1}, z_{r2}) by the same element of $SU(2)$. The orbit thereby generated is a manifold diffeomorphic to $SU(2)$, as indicated in figure 5. The projection of this manifold onto angular momentum space is the surface $SO(3)$ shown in the figure, that is, all orientations of the triangle are generated. For example, in the 3-torus T_B over the angular momentum triangle with orientation B, there is a point b that can be reached from the given point a by some spinor rotation. The spinor rotation in question is one of the two that projects onto the $SO(3)$ rotation that maps A into B, according to (21). The orbit of reference point a under the $SU(2)$ action therefore passes through the 3-tori over every possible orientation of the triangle. In fact, it passes through each 3-torus twice, since the $SU(2)$ rotation $u = -1$ is just a phase factor. This is the meaning of points a' and b' in the figure, which are related to points a and b by multiplying all three spinors by -1.

Thus any point on the Wigner manifold can be reached from the reference point a by applying some $SU(2)$ rotation, and then adjusting the overall phases of the three spinors (z_{r1}, z_{r2}). The first step is equivalent to following along the Hamiltonian flows in the large phase space generated by the three J_i (this creates the $SU(2)$ rotation), while the second is equivalent to following the Hamiltonian flows generated by the three I_r (this changes the overall phases of the three spinors). By letting the rotations range over all of $SU(2)$ and the three angles ψ_r range from 0 to 4π, the Wigner manifold is covered twice. Thus we obtain coordinates on the Wigner manifold $(\alpha, \beta, \gamma, \psi_1, \psi_2, \psi_3)$ (the first three of which are Euler angles on $SU(2)$).

Solving the simultaneous amplitude transport equations for the six observables $I_r, J_i, r, i = 1, 2, 3$, requires us to find an invariant measure on the Wigner manifold, that is, one invariant under all of the corresponding Hamiltonian flows. Details are presented in section 11; for now we just guess that this measure is the Haar measure on the group $SU(2)$ times the obvious measure on the 3-tori generated by I_r, namely,

$$\sin\beta\,\mathrm{d}\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\gamma \wedge \mathrm{d}\psi_1 \wedge \mathrm{d}\psi_2 \wedge \mathrm{d}\psi_3, \tag{45}$$

where (α, β, γ) are Euler angles on $SU(2)$. The integral of this measure over the Wigner manifold is

$$V_W = \tfrac{1}{2}(16\pi^2)(4\pi)^3 = 2^9\pi^5, \tag{46}$$

where $1/2$ compensates for the fact that the Wigner manifold is covered twice when the Euler angles run over $SU(2)$ and each ψ_r runs from 0 to 4π.

7.2. *Angles related to the shape of the triangle*

Figure 4 defines the angles η_r as the angles opposite vectors $\mathbf{J}_r$. Under our assumptions, these angles lie in the range $0 \leqslant \eta_r \leqslant \pi$. By projecting all three vectors onto the directions parallel and orthogonal to each of the vectors in turn, we obtain a series of identities,

$$j_1 \cos\eta_2 + j_2 \cos\eta_1 + j_3 = 0, \tag{47}$$

$$j_1 \sin\eta_2 - j_2 \sin\eta_1 = 0, \tag{48}$$

and four more obtained by cycling indices 1, 2, 3. These allow us to solve for the cosines of the angles η_r,

$$\cos\eta_1 = \frac{j_1^2 - j_2^2 - j_3^2}{2j_2 j_3}, \tag{49}$$

and cyclic permutations, which in view of the stated ranges on the angles allows all three angles η_r to be uniquely determined as functions of (j_1, j_2, j_3). We shall regard angles η_r as convenient substitutions for these definite functions of the lengths of the angular momentum vectors.

For the stated ranges on η_r, the sines of the angles are nonnegative and are related to the area Δ of the triangle, as follows:

$$\Delta = \frac{1}{2}|\mathbf{J}_1 \times \mathbf{J}_2| = \frac{1}{2} j_1 j_2 \sin\eta_3$$
$$= \frac{1}{4}\sqrt{(j_1+j_2+j_3)(-j_1+j_2+j_3)(j_1-j_2+j_3)(j_1+j_2-j_3)}, \tag{50}$$

and cyclic permutations. Some authors define Δ as the final square root (without the 1/4).

8. Intersections of manifolds

The stationary phase points of the matrix element (22) are the intersections of the jm-manifold and the Wigner manifold in the large phase space. Thus we must use a version of (5) for the matrix element, rather than (3). In this section we study the intersections of the manifolds, continuing with a classical picture.

If the jm-torus and the Wigner manifold in the large phase space have a common point of intersection, then the projections of these two manifolds onto angular momentum space must have a common point of intersection. The converse is also true: if the projections have a point in common, then the inverse image of this point under π, a 3-torus which is the orbit of the three I_r-flows, must contain two points, one of which belongs to the jm-torus and the other to the Wigner manifold. But the 3-torus is the orbit of the I_r-flows, and these flows confine one to both the jm-torus and the Wigner manifold. Therefore, the entire 3-torus is common to both the jm-torus and the Wigner manifold. Therefore, to find intersections of the jm-torus and the Wigner manifold, we may first find the intersections of their projections under π.

The jm-torus projects onto a set of configurations of three angular momenta with given lengths and fixed values of $m_r = J_{rz}$, with arbitrary azimuthal angles, while the Wigner manifold projects onto configurations with the same lengths in which the vector sum of the angular momenta vanishes, forming a triangle, with arbitrary orientation. Therefore, to find a common point between these two sets of classical angular momenta configurations, we can either adjust the azimuthal angles of the angular momenta with given m values until a triangle is formed (total $\mathbf{J} = 0$) or we can rotate a triangle from a given, reference orientation until the m values are the desired ones. We choose the latter procedure.

Our reference orientation of the triangle is shown in figure 4, which is indicated schematically as the point A in figure 5. We must rotate this reference orientation to obtain some prescribed values of m_r. These values satisfy relations (40), so in particular $|m_3| \leqslant j_3$. Thus by rotating the triangle in the reference orientation about the y-axis by a unique angle β, $0 \leqslant \beta \leqslant \pi$, defined by

$$m_3 = j_3 \cos\beta, \tag{51}$$

we guarantee that $\mathbf{J}_3$ has the right projection. The result of this rotation is shown in figure 6, for a certain negative value of m_3.

Next we rotate the triangle about the axis $\mathbf{J}_3$ by an angle γ, which does not change $\mathbf{J}_3$ or its projection, but which rotates $\mathbf{J}_1$ and $\mathbf{J}_2$ in a cone, as illustrated in figure 7. We wish to choose the angle γ so that $\mathbf{J}_2$ has the desired projection m_2 onto the z-axis. In figure 7, $\mathbf{J}_1$ is not shown, but $\mathbf{J}_2$ rotates about the $\mathbf{J}_3$ direction, its tip sweeping out circle C_3. The circle C_z in the figure is swept out by a vector of length j_2 and projection m_2 onto the z-axis (this vector is not shown). Circles C_3 and C_z intersect in two points Q and Q' in the figure, which represent two orientations of the triangle that have the correct values of both m_3 and m_2. Now the orientation of the triangle is fixed, so there is no more freedom to rotate $\mathbf{J}_1$. In this final orientation the value of J_{1z} is $-J_{2z} - J_{3z} = -m_2 - m_3$, since $\mathbf{J} = 0$ for the triangle. Either this value of J_{1z} equals the value of m_1 associated with the jm-torus or it does not. If it does not,

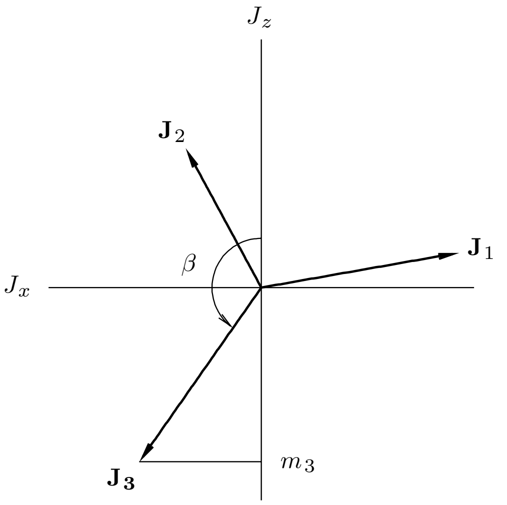

Figure 6. By rotating the reference orientation of the triangle about the y-axis, we can give $\mathbf{J}_3$ the desired projection m_3 onto the z-axis.

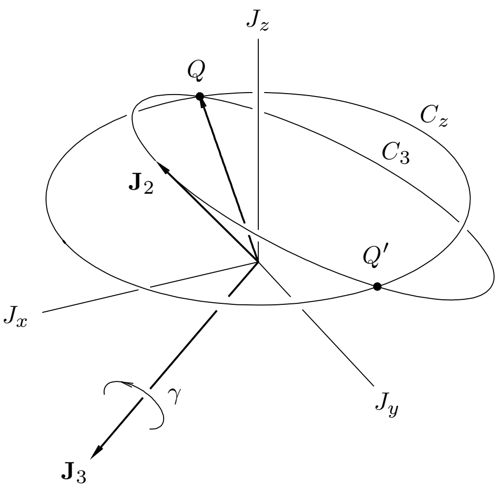

Figure 7. Once vector $\mathbf{J}_3$ has the desired projection m_3, we rotate the triangle by angle γ about the axis $\mathbf{J}_3$ to make $\mathbf{J}_2$ have its desired projection m_2. This cannot always be done for real angles γ, but when it can be done there are generically two angles that work, illustrated by points Q and Q' in the figure.

then there are no intersections between the jm-torus and the Wigner manifold. This is just the classical expression of the condition that the matrix element (22) vanishes unless $\sum_r m_r = 0$. Henceforth we assume that the m_r values for the jm-torus do satisfy this condition.

In this case we may solve for the values of γ associated with points Q and Q' in figure 7. Writing $R(\mathbf{n}, \theta)$ for a three-dimensional rotation by angle θ about axis $\mathbf{n}$, we have applied the rotation

$$R(\mathbf{j}_3, \gamma)R(\mathbf{y}, \beta) = R(\mathbf{y}, \beta)R(\mathbf{z}, \gamma) \tag{52}$$

to the reference orientation in figure 4, where $\mathbf{j}_3$ is the unit vector in the direction $\mathbf{J}_3$ shown in figure 6 (after the first rotation). In the reference orientation the vectors are

$$\mathbf{J}_1 = j_1 \begin{pmatrix} \sin\eta_2 \\ 0 \\ \cos\eta_2 \end{pmatrix}, \qquad \mathbf{J}_2 = j_2 \begin{pmatrix} -\sin\eta_1 \\ 0 \\ \cos\eta_1 \end{pmatrix}, \qquad \mathbf{J}_3 = j_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{53}$$

After applying rotation (52) these become

$$\begin{aligned} \mathbf{J}_1 &= j_1 \begin{pmatrix} \cos\beta\cos\gamma\sin\eta_2 + \sin\beta\cos\eta_2 \\ \sin\gamma\sin\eta_2 \\ -\sin\beta\cos\gamma\sin\eta_2 + \cos\beta\cos\eta_2 \end{pmatrix}, \\ \mathbf{J}_2 &= j_2 \begin{pmatrix} -\cos\beta\cos\gamma\sin\eta_1 + \sin\beta\cos\eta_1 \\ -\sin\gamma\sin\eta_1 \\ \sin\beta\cos\gamma\sin\eta_1 + \cos\beta\cos\eta_1 \end{pmatrix}, \\ \mathbf{J}_3 &= j_3 \begin{pmatrix} \sin\beta \\ 0 \\ \cos\beta \end{pmatrix}. \end{aligned} \tag{54}$$

We have already solved for β in (51); we may now solve for γ by demanding either $J_{1z} = m_1$ or $J_{2z} = m_2$. These lead to

$$\cos\gamma = \frac{j_1\cos\beta\cos\eta_2 - m_1}{j_1\sin\beta\sin\eta_2} = \frac{m_2 - j_2\cos\beta\cos\eta_1}{j_2\sin\beta\sin\eta_1}. \tag{55}$$

These two conditions are equivalent (under the assumption $\sum_r m_r = 0$), as follows from the identities (47)–(49). If the common value of the two expressions on the right-hand side of (55) lies in the range $(-1, +1)$, then there are two real angles γ satisfying (55), corresponding to the two points Q and Q' in figure 7. In this case the two manifolds have real intersections, and we are in the classically allowed region for the $3j$-symbol. We let γ represent the root (the 'principal branch') in the range $[0, \pi]$, and $-\gamma$ the root (the 'secondary branch') in the range $[-\pi, 0]$. Note that $\sin\gamma \geqslant 0$ ($\leqslant 0$) on the principal (secondary) branch. If the right-hand side of (55) lies outside the range $[-1, 1]$, then there are two complex roots for γ. In this case the two manifolds have no real intersections, but they do have complex ones. Only one of the two complex roots is picked up by the contour of integration used in obtaining the matrix element (3) or (5), resulting in an exponentially decaying expression for the matrix element. In this case we are in the classically forbidden region of the $3j$-symbol. In the following for simplicity we assume we are in the classically allowed region.

The points Q and Q' in figure 7 represent values of $\mathbf{J}_2$ in a single angular momentum space $\mathbb{R}^3$. Taken with the values of $\mathbf{J}_1$ and $\mathbf{J}_3$, they specify points, call them P and P', in the combined angular momentum space $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ that lie on the intersection of the projections of the jm-torus and the Wigner manifold onto that space. Then by applying rotations about the z-axis to P and P', we generate a pair of circles in angular momentum space. Such rotations change neither the z-projections of the three vectors nor their vector sum (zero). This is obviously a reflection of the fact that the operator J_z defining the state on the right-hand side of (22) is a function of the operators (J_{1z}, J_{2z}, J_{3z}) defining the state on the left. Thus, the projections of the jm-manifold and Wigner manifold under π intersect generically in a pair of circles.

Thus, the intersection of the jm-torus and the Wigner manifold in the large phase space is the inverse image of this pair of circles under π, generically a pair of 3-torus bundles over a circle. Since the I_r-flows and the J_z-flow commute, these bundles are trivial, in fact each is a 4-torus, on which coordinates are $(\psi_1, \psi_2, \psi_3, \phi)$, where ϕ is the angle of evolution along the

J_z-flow. The volume of either one of the 4-tori with respect to the measure $d\psi_1 \wedge d\psi_2 \wedge d\psi_3 \wedge d\phi$ is

$$V_I = \tfrac{1}{2}(4\pi)^4, \tag{56}$$

the factor of 1/2 being explained by figure 1.

The jm-torus and the Wigner manifold, both six dimensional, intersect in a 4-torus because the lists of functions defining the two manifolds, $(I_1, I_2, I_3, J_{1z}, J_{2z}, J_{3z})$ and $(I_1, I_2, I_3, J_x, J_y, J_z)$, have three functions in common while J_z in the second list is a function of (J_{1z}, J_{2z}, J_{3z}) in the first list. Below we will transform the functions to make both lists have explicitly four variables in common (see equation (97)).

9. Action integrals

Action integrals on the jm-torus and the Wigner manifold are needed for the phases in expressions such as (5). We only need the action function at some point on the intersection between the two manifolds, which gives us a lot of choice since that intersection is a 4-torus. We continue with a classical picture in this section.

9.1. Choosing reference points

Action integrals are defined relative to some initial or reference point on each manifold. For the jm-torus, a convenient point is the one where $\theta_{r\mu} = 0$, $r = 1, 2, 3$, $\mu = 1, 2$, that is, the point where each $z_{r\mu}$ is real and nonnegative, as explained in section 6. According to (39), the spinors at this reference point are given explicitly by

$$\begin{pmatrix} z_{r1} \\ z_{r2} \end{pmatrix} = \begin{pmatrix} \sqrt{j_r + m_r} \\ \sqrt{j_r - m_r} \end{pmatrix}. \tag{57}$$

The projection of this point onto angular momentum space is a set of vectors $\mathbf{J}_r$, $r = 1, 2, 3$, of given lengths j_r that lie in the x–z plane, with $J_{rz} = m_r$ and $J_x \geqslant 0$, as shown by (34)–(36). Such vectors are illustrated in figure 3.

As for the Wigner manifold, it is convenient to take the reference point to be point a in figure 5, which is discussed in section 7. This point projects onto the standard orientation of the triangle, point A in figure 5, where the angular momentum vectors have the values shown in (53). At the point a, z_{r1} is real and positive for all r, as explained in section 7. For example, for $r = 1$ this assumption combined with (44) implies $z_{11} = \sqrt{j_1 + J_{1z}}$, which by (53) becomes $z_{11} = \sqrt{j_1(1 + \cos\eta_2)} = \sqrt{2j_1}\cos\eta_2/2$. Then (35) and $J_{1y} = 0$ imply that z_{12} is purely real, and (34) allows us to solve for z_{12} in terms of J_{1x}, given by (53), producing finally $z_{12} = \sqrt{2j_1}\sin\eta_2$. Proceeding similarly with the other two spinors $r = 2, 3$, we obtain the three spinors at the reference point a on the Wigner manifold,

$$\begin{pmatrix} z_{11} \\ z_{12} \end{pmatrix} = \sqrt{2j_1}\begin{pmatrix} \cos\eta_2/2 \\ \sin\eta_2/2 \end{pmatrix}, \qquad \begin{pmatrix} z_{21} \\ z_{22} \end{pmatrix} = \sqrt{2j_2}\begin{pmatrix} \cos\eta_1/2 \\ -\sin\eta_1/2 \end{pmatrix}, \qquad \begin{pmatrix} z_{31} \\ z_{32} \end{pmatrix} = \sqrt{2j_3}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{58}$$

Now to obtain a point common to both the jm-torus and the Wigner manifold, we apply the spinor rotation

$$u(\mathbf{y}, \beta)u(\mathbf{z}, \gamma) = \begin{pmatrix} e^{-i\gamma/2}\cos\beta/2 & -e^{i\gamma/2}\sin\beta/2 \\ e^{-i\gamma/2}\sin\beta/2 & e^{i\gamma/2}\cos\beta/2 \end{pmatrix} \tag{59}$$

to the reference spinors (z_{r1}, z_{r2}), $r = 1, 2, 3$, in (58), where Euler angles β and γ are defined by (51) and (55). We obtain either the principal branch or the secondary one by taking $\gamma \geqslant 0$ or $\gamma \leqslant 0$, respectively. The spinor rotation (59) induces the 3×3 rotation on the angular momentum vectors shown in (52). Thus we obtain the spinors at the common point of intersection between the jm-torus and the Wigner manifold,

$$\begin{pmatrix} z_{11} \\ z_{12} \end{pmatrix} = \sqrt{2j_1} \begin{pmatrix} e^{-i\gamma/2}\cos\beta/2\cos\eta_2/2 - e^{i\gamma/2}\sin\beta/2\sin\eta_2/2 \\ e^{-i\gamma/2}\sin\beta/2\cos\eta_2/2 + e^{i\gamma/2}\cos\beta/2\sin\eta_2/2 \end{pmatrix}, \tag{60}$$

$$\begin{pmatrix} z_{21} \\ z_{22} \end{pmatrix} = \sqrt{2j_2} \begin{pmatrix} e^{-i\gamma/2}\cos\beta/2\cos\eta_1/2 + e^{i\gamma/2}\sin\beta/2\sin\eta_1/2 \\ e^{-i\gamma/2}\sin\beta/2\cos\eta_1/2 - e^{i\gamma/2}\cos\beta/2\sin\eta_1/2 \end{pmatrix}, \tag{61}$$

$$\begin{pmatrix} z_{31} \\ z_{32} \end{pmatrix} = e^{-i\gamma/2}\sqrt{2j_3} \begin{pmatrix} \cos\beta/2 \\ \sin\beta/2 \end{pmatrix}. \tag{62}$$

One can easily check using (34)–(36) that these spinors project onto the angular momentum vectors in (54).

9.2. Computing the actions

In computing action integrals we use the identity,

$$\sum_{r\mu} p_{r\mu}\, dx_{r\mu} = \frac{i}{2}\sum_{r\mu}(\bar{z}_{r\mu}\, dz_{r\mu} - z_{r\mu}\, d\bar{z}_{r\mu}) + \frac{1}{2}\, d\sum_{r\mu} x_{r\mu} p_{r\mu}. \tag{63}$$

The integral of the left-hand side is the usual action one would need for wavefunctions $\psi(x_{11}, \ldots, x_{32})$, but it can be replaced by the integral of the first differential form on the right, for the following reason. First, the integral of the exact differential on the right contributes the difference in the function $(1/2)\sum_{r\mu} x_{r\mu} p_{r\mu}$ between the initial and final points. But the final point is the common point of intersection between the jm-torus and the Wigner manifold, so this contribution cancels when we subtract actions as in (5). As for the initial points on the two manifolds, these have been chosen (see equations (57) and (58)) so that all $z_{r\mu}$ are purely real or $p_{r\mu} = 0$. Thus the function in question vanishes at the initial points. As for the integral of the first term on the right-hand side of (63), it can be written as

$$S = \operatorname{Im}\int \sum_{r\mu} z_{r\mu}\, d\bar{z}_{r\mu}. \tag{64}$$

For the action on the jm-torus between initial point (57) and final point (60)–(62), we follow a path consisting of flows of the functions $I_{r\mu} = (1/2)|z_{r\mu}|^2$ taken one at a time by angles $\theta_{r\mu}$. Along the $I_{r\mu}$-flow we have $d\bar{z}_{r\mu}/d\theta_{r\mu} = (i/2)\bar{z}_{r\mu}$, so the contribution to S is

$$\operatorname{Im}\int_0^{\theta_{r\mu}} \frac{i}{2}|z_{r\mu}|^2 d\theta_{r\mu} = I_{r\mu}\theta_{r\mu}, \tag{65}$$

since $I_{r\mu}$ is constant along its own flow and since $\theta_{r\mu} = 0$ at the reference point. Thus the total action between initial and final points on the jm-torus is

$$S_{jm} = \sum_{r\mu} I_{r\mu}\theta_{r\mu}. \tag{66}$$

Under the canonical transformation (41) this becomes

$$S_{jm} = \sum_r (I_r\psi_r + J_{rz}\phi_r) = \sum_r (j_r\psi_r + m_r\phi_r), \tag{67}$$

where in the final form we replace I_r and J_{rz} by their values on a given jm-torus.

The angles $\theta_{r\mu}$ or (ψ_r, ϕ_r) are the coordinates of the final point specified by equations (60)–(62). The solutions of Hamilton's equations for the I_r-flow can be written as $z_{r\mu}(\theta_{r\mu}) = z_{r\mu}(0)\exp(-\mathrm{i}\theta_{r\mu}/2)$ (see figure 2) where the initial conditions are real and nonnegative, so we have $\theta_{r\mu} = 2\arg\bar{z}_{r\mu}$. Combining this and (42), we can write the action on the jm-torus as

$$S_{jm} = 2\sum_{r\mu} I_{r\mu}\arg\bar{z}_{r\mu} = \sum_{r} j_r \arg(\bar{z}_{r1}\bar{z}_{r2}) + \sum_{r} m_r \arg(\bar{z}_{r1}z_{r2}). \tag{68}$$

Using equations (60)–(62), this can be written as

$$\begin{aligned} S_{jm} = j_3\gamma &+ j_1\arg(\cos\beta\sin\eta_2 + \sin\beta\cos\gamma\cos\eta_2 + \mathrm{i}\sin\beta\sin\gamma) \\ &+ j_2\arg(-\cos\beta\sin\eta_1 + \sin\beta\cos\gamma\cos\eta_1 + \mathrm{i}\sin\beta\sin\gamma) \\ &+ m_1\arg(\sin\beta\cos\eta_2 + \cos\beta\cos\gamma\sin\eta_2 + \mathrm{i}\sin\gamma\sin\eta_2) \\ &+ m_2\arg(\sin\beta\cos\eta_1 - \cos\beta\cos\gamma\sin\eta_1 - \mathrm{i}\sin\gamma\sin\eta_1). \end{aligned} \tag{69}$$

Here we have used the rule $\arg(ab) = \arg a + \arg b$, which is only valid for certain choices of branch of the arg function. A more careful analysis shows that (69) is the correct action along a certain path from the initial to final point (the principal branch) on the jm-torus if the range of the arg function is taken to be $[-\pi, \pi)$. (The path is defined by $\gamma \leqslant \theta_{11}, \theta_{22} \leqslant 2\pi$, $-\gamma \leqslant \theta_{12}, \theta_{21} \leqslant \gamma$, that is, one integrates from 0 to these final θ values.) In particular, this means that $\psi_1, \psi_2, \psi_3, \phi_1$ all lie in $[0, \pi]$, while ϕ_2 lies in $[-\pi, 0]$. These ranges on angles ϕ_1, ϕ_2 are also evident from figure 7. Similarly, for the secondary branch ($-\pi \leqslant \gamma \leqslant 0$, $\sin\gamma \leqslant 0$) there exists a path such that with the same range on the arg function (69) is still correct. With these understandings, the values of S_{jm} on the two branches differ by a sign. We shall henceforth write S_{jm} ($-S_{jm}$) for the principal (secondary) branch.

Equation (66) can also be written in terms of $\cos^{-1}$ functions. We note that $\arg(\bar{z}_{11}\bar{z}_{12}) = \cos^{-1}[\mathrm{Re}(\bar{z}_{11}\bar{z}_{12})/|z_{11}z_{12}|]$ and that $|z_{11}z_{12}| = \sqrt{j_1^2 - m_1^2}$, etc. We can also use (55) to eliminate $\cos\gamma$. For the principal branch ($\gamma \geqslant 0$) this gives

$$\begin{aligned} S_{jm} = j_1\cos^{-1}&\left(\frac{j_1\cos\beta - m_1\cos\eta_2}{\sin\eta_2 J_{1\perp}}\right) + j_2\cos^{-1}\left(\frac{m_2\cos\eta_1 - j_2\cos\beta}{\sin\eta_1 J_{2\perp}}\right) \\ &+ j_3\cos^{-1}\left(\frac{j_1\cos\beta\cos\eta_2 - m_1}{j_1\sin\beta\sin\eta_2}\right) + m_1\cos^{-1}\left(\frac{j_1\cos\eta_2 - m_1\cos\beta}{\sin\beta J_{1\perp}}\right) \\ &- m_2\cos^{-1}\left(\frac{j_2\cos\eta_1 - m_2\cos\beta}{\sin\beta J_{2\perp}}\right), \end{aligned} \tag{70}$$

where

$$J_{r\perp} = \sqrt{j_r^2 - m_r^2}, \tag{71}$$

and where the range of the $\cos^{-1}$ function is $[0, \pi]$. Finally, by using equations (49) and (50) these can be written explicitly in terms of the parameters j_r, m_r. The result has the form of (67), where

$$\psi_1 = \cos^{-1}\left(\frac{j_1^2(m_3 - m_2) + m_1(j_3^2 - j_2^2)}{4\Delta J_{1\perp}}\right), \tag{72}$$

and cyclic permutations of indices, and where

$$\phi_1 = \cos^{-1}\left(\frac{j_2^2 - j_3^2 - j_1^2 - 2m_1m_3}{2J_{1\perp}J_{3\perp}}\right), \qquad \phi_2 = -\cos^{-1}\left(\frac{j_1^2 - j_3^2 - j_2^2 - 2m_2m_3}{2J_{2\perp}J_{3\perp}}\right), \tag{73}$$

and where $\phi_3 = 0$.

Now we consider the action on the Wigner manifold between the initial point (58) and the final point (60)–(62). The path between these points is made up of the product of rotations (59), so we consider the action integral (64) along a rotation by angle θ generated by $\mathbf{n}\cdot\mathbf{J}$. Hamilton's equations (see equation (28)) are $\mathrm{d}\bar{z}_{r\mu}/\mathrm{d}\theta = (\mathrm{i}/2)\sum_\nu \bar{z}_{r\nu}(\mathbf{n}\cdot\boldsymbol{\sigma})_{\nu\mu}$, so by (64) we have

$$S = \mathrm{Im}\int_0^\theta \frac{\mathrm{i}}{2}\sum_{r\mu\nu}\bar{z}_{r\nu}(\mathbf{n}\cdot\boldsymbol{\sigma})_{\nu\mu}z_{r\mu}\,\mathrm{d}\theta = \int_0^\theta(\mathbf{n}\cdot\mathbf{J})\,\mathrm{d}\theta = (\mathbf{n}\cdot\mathbf{J})\theta = 0, \tag{74}$$

where we use (33), the fact that $\mathbf{n}\cdot\mathbf{J}$ is constant along its own flow, and the fact that $\mathbf{J} = 0$ on the Wigner manifold. The rotational action vanishes.

Thus the phase of the matrix element (22) is determined entirely by the action integral along the jm-torus, that is, to within a sign it is given by equations (66)–(73). This is the phase function determined previously by Ponzano and Regge, Miller, and others, and we see that it is essentially a simple combination of the phases of the Schwinger oscillators. We have, however, determined this phase function entirely within a classical model, that is, without imposing any quantization conditions on the manifolds.

10. Bohr–Sommerfeld quantization

We do not need Bohr–Sommerfeld approximations to the eigenvalues of the operators involved in the $3j$-symbols because those eigenvalues are known exactly. We must, however, quantize the jm-torus and the Wigner manifold, to obtain the wavefunctions whose scalar product is the $3j$-symbol. We also need the Bohr–Sommerfeld rules to make the connection between the contour values for various classical functions and the standard quantum numbers of the associated operators.

To quantize a Lagrangian manifold we must first find the generators of the fundamental or first homotopy group of the manifold, that is, a set of closed contours in terms of which all closed contours can be generated by concatening curves. In the following we shall call these generators 'basis contours', although technically the fundamental group, even when Abelian, is a group and not a vector space. For example, in the familiar case of the invariant n-tori of integrable systems of n degrees of freedom, the fundamental group is $\mathbb{Z}^n$, that is, an arbitrary closed contour is expressed as a 'linear combination' of the n basis contours with integer coefficients. The n basis contours themselves go around the torus once in the n different directions.

After finding the basis contours, we compute the total phase associated with each of them, the sum of an action, the integral of $p\,\mathrm{d}q$ around the contour and a Maslov phase, which is $-\pi/2$ times the Maslov index of the loop. Both these phases are topological invariants and are additive when loops are concatenated. Then we demand that the total phase be a multiple of 2π; this is the consistency condition on the semiclassical wavefunction that selects out certain manifolds as being 'quantized'.

The Lagrangian manifolds we are interested in are level sets of a set of classical functions that are the principal symbols of a set of operators, which in our application need not commute. Quantized Lagrangian manifolds support wavefunctions that are approximate eigenfunctions of the set of operators. The corresponding eigenvalues are the contour values of the principal symbols, to within errors of order $\hbar^2$.

10.1. Quantizing the jm-tori

In the case of the jm-tori, whose fundamental group is $\mathbb{Z}^6$, we are dealing with the eigenfunctions of a set of independent harmonic oscillators, so the problem could not be more elementary from a semiclassical standpoint. There are, however, interesting issues that arise. Let the complete set of commuting quantum observables be $(\hat{I}_r, \hat{J}_{rz}), r = 1, 2, 3$, defined in (8) and (9). Let us denote the Weyl symbol of an operator $\hat{A}$ by sym$(\hat{A})$. Then we have

$$\mathrm{sym}(\hat{I}_r) = I_r - \tfrac{1}{2}, \qquad \mathrm{sym}(\hat{J}_{rz}) = J_{rz}, \tag{75}$$

where I_r and J_{rz} (the classical functions) are defined by equations (32) and (33). The operator $\hat{I}_r$ violates our assumption in section 2 that the commuting operators defining our integrable system should have Weyl symbols that are even power series in $\hbar$ (since $-1/2$ is of order $\hbar$). Our assumption is valid for the harmonic oscillators $\hat{H}_r = (1/2)\sum_\mu \left(\hat{x}_{r\mu}^2 + \hat{p}_{r\mu}^2\right)$, but in defining $\hat{I}_r$ in (8) we have subtracted the zero point energy, a constant of order $\hbar$, $\hat{I}_r = (1/2)(\hat{H}_r - 1)$, so that the eigenvalues of $\hat{I}_r$ would be the conventional quantum numbers j_r for an angular momentum, and so that the identity (13) would have a familiar form. In the following we shall take the principal symbol of $\hat{I}_r$ to be the whole symbol, including $-1/2$. This achieves the same results we would have had if we had worked with $\hat{H}_r$ instead of $\hat{I}_r$ and defined the principal symbol as the leading term in $\hbar$, as in section 2, since sym$(\hat{H}_r) = 2I_r$. We must be careful, however, since the principal symbol of $\hat{I}_r$ is not I_r. For most of the other operators we shall use, the principal symbol is obtained simply by removing the hat (for example, J_{rz} above).

The basis contours on the jm-torus are most easily expressed as the contours on which each of the angles $\theta_{r\mu}$ is allowed to go from 0 to 4π while all other $\theta_{r\mu}$'s are held fixed. We may also use any linear combination of these contours with integer coefficients and unit determinant. In terms of the angles ψ_r, ϕ_r, given by (42), a convenient choice is to take one basis contour as the path on which one ψ_r goes from 0 to 4π while all others ψ_r's and all ϕ_r's are held fixed; this is following the I_r-flow for elapsed angle $\psi_r = 4\pi$. A second basis contour may be taken to be the path on which ψ_r goes from 0 to 2π, and then ϕ_r goes from 0 to 2π; the two legs involve following the I_r-flow and then the J_{rz}-flow, each for elapsed angle 2π. Doing this for $r = 1, 2, 3$ gives us six basis contours on the jm-torus.

The action along the first basis contour is computed as in (65). Hamilton's equations for I_r are $\mathrm{d}\bar{z}_{r\mu}/\mathrm{d}\psi_r = (\mathrm{i}/2)\bar{z}_{r\mu}$, so we obtain $S_{r1} = 4\pi I_r$, where S_{r1} refers to the action along the first basis contour. For the second basis contour, the first leg contributes an action $2\pi I_r$, while the second leg, which follows the flow generated by J_{rz}, is a rotation whose action may be computed as in (74), but with $\mathbf{J}$ replaced by $\mathbf{J}_r$ since we do not sum over r. Thus the final answer does not vanish ($\mathbf{J}_r$ is nonzero on the jm-torus), and the contribution from the second leg is $2\pi J_{rz}$ or $S_{r2} = 2\pi(I_r + J_{rz})$. Altogether, we have

$$S_{r1} = 4\pi j_r, \qquad S_{r2} = 2\pi(j_r + m_r), \tag{76}$$

where 1 and 2 refer to the first and second basis contours associated with a particular value of r, and where we have replaced I_r and J_{rz} by their contour values j_r and m_r on the jm-torus.

Next we need the Maslov indices along the two basis contours. Here we follow the computational method described in Littlejohn and Robbins (1987), which uses the determinant of complex matrices and which is based ultimately on Arnold (1967). Similar techniques are discussed by Mishchenko *et al* (1990). The method works for finding Maslov indices along closed curves on orientable Lagrangian manifolds in $\mathbb{R}^{2n}$. To describe the method we adopt a general notation, in which global coordinates on phase space are $(q_1, \ldots, q_n, p_1, \ldots, p_n)$. We suppose that there exists a set of n vector fields on the Lagrangian manifold, linearly independent at each point, so that they span the Lagrangian tangent plane at each point. In our applications, these are the Hamiltonian vector fields associated with a set of functions $(A_1, \ldots, A_n)$. We consider the rate of change of the quantities $q_i - \mathrm{i}p_i$ along the jth vector field, which is the Poisson bracket $\{q_i - \mathrm{i}p_i, A_j\}$. The set of these Poisson brackets forms an $n \times n$ complex matrix M_{ij} that is never singular, so $\det M$ traces out a closed loop in the complex plane without passing through the origin when we go around a closed loop on the Lagrangian manifold. Then the Maslov index μ associated with this loop is given by

$$\mu = 2\,\mathrm{wn}\det M, \tag{77}$$

where wn refers to the winding number of the loop in the complex plane, reckoned as positive in the counterclockwise direction. The winding number is invariant when M_{ij} is multiplied by any nonzero complex constant (or constant matrix), so such constants can be dropped in the calculation.

For the jm-torus, we identify q's and p's with the coordinates $x_{r\mu}$ and $p_{r\mu}$, and A's with the functions $(I_1, I_2, I_3, J_{1z}, J_{2z}, J_{3z})$. Then we can replace $q_i - \mathrm{i}p_i$ by $\bar{z}_{r\mu}$, dropping the $1/\sqrt{2}$. The needed matrix elements are

$$\begin{aligned}
\{\bar{z}_{r\mu}, I_s\} &= \mathrm{i}\frac{\partial I_s}{\partial z_{r\mu}} = \frac{\mathrm{i}}{2}\delta_{rs}\bar{z}_{r\mu},\\
\{\bar{z}_{r1}, J_{sz}\} &= \mathrm{i}\frac{\partial J_{sz}}{\partial z_{r1}} = \frac{\mathrm{i}}{2}\delta_{rs}\bar{z}_{r1},\\
\{\bar{z}_{r2}, J_{sz}\} &= \mathrm{i}\frac{\partial J_{sz}}{\partial z_{r2}} = -\frac{\mathrm{i}}{2}\delta_{rs}\bar{z}_{r2}.
\end{aligned} \tag{78}$$

We drop the constant $\mathrm{i}/2$ on the right-hand side, and choose the ordering $(I_1, J_{1z}, I_2, J_{2z}, I_3, J_{3z})$ for the functions. Then the 6×6 matrix block diagonalizes into three 2×2 blocks, and we find

$$\det M = \bar{z}_{11}\bar{z}_{12}\bar{z}_{21}\bar{z}_{22}\bar{z}_{31}\bar{z}_{32}, \tag{79}$$

to within a constant.

Along the flow of one of I_r's we have $\bar{z}_{r\mu}(\psi_r) = \exp(\mathrm{i}\psi_r/2)\bar{z}_{r\mu}(0)$ or $\det M(\psi_r) = \exp(\mathrm{i}\psi_r)\det M(0)$. Therefore when the given ψ_r goes from 0 to 4π, the other ψ_r's being held fixed (this is the first basis contour), $\det M$ circles the origin twice and we have $\mu = 4$. Along the flow of one of J_{rz}'s, however, we have $\bar{z}_{r1}(\phi_r) = \exp(\mathrm{i}\phi_r/2)\bar{z}_{r1}(0)$ and $\bar{z}_{r2}(\phi_r) = \exp(-\mathrm{i}\phi_r/2)\bar{z}_{r2}(0)$ or $\det M(\phi_r) = \det M(0)$. Therefore along the second basis contour the I_r-flow takes $\det M$ once around the origin (elapsed parameter $\psi_r = 2\pi$), while the J_{rz}-flow does nothing. Therefore the Maslov index of the second basis contour is $\mu = 2$. There are easier ways to find the Maslov indices of harmonic oscillators, but this calculation is useful practice for the case of the Wigner manifold that we take up momentarily.

Now we apply the quantization conditions. For the first basis contour the total phase is $4\pi j_r - 4(\pi/2)$, which we set to $2n_r\pi$ where n_r is an integer. Thus the quantized tori must satisfy $j_r = (n_r + 1)/2$. The allowed values of n_r are determined by the fact $n_r < -1$ is impossible in view of the fact that I_r is nonnegative definite, and $n_r = -1$ corresponds to a torus of less than full dimensionality (six), so the wavefunction (1) is not meaningful. Thus we must have $n_r = 0, 1, \ldots$. The j_r in these formulae, and throughout all of the classical analysis from section 5 up to this point, has referred to a contour value for the function I_r; the only difference now is that we are restricting the value of j_r in order that the torus be quantized. This j_r, however, is not value of the principal symbol of the operator $\hat{I}_r$ (see equation (75)), so the Bohr–Sommerfeld or EBK quantization rule gives the semiclassical eigenvalue of $\hat{I}_r$, call it j_r^{qu}, as

$$j_r^{\mathrm{qu}} = j_r - \frac{1}{2} = \frac{n_r}{2}. \tag{80}$$

The semiclassical eigenvalues of $\hat{I}_r$ are nonnegative integers or half-integers, the exact answer (not surprising in view of the fact that semiclassical quantization of quadratic Hamiltonians is exact). If we use the operator identity (13) to find the eigenvalues of operators $\mathbf{J}_r^2$, these are also exact.

Equation (80) shows that the classical level set corresponding to quantum number j_r^{qu} is $j_r = j_r^{\text{qu}} + 1/2$. The extra 1/2 in this formula has caused some discussion in the past and merits a little more now. Ponzano and Regge (1968) used intuition and numerical evidence to argue for the presence of 1/2. Miller, without knowing about Ponzano and Regge, also included 1/2, referring to the 'usual' semiclassical replacement for angular momenta. Presumably he was referring to the similar replacement that occurs in the treatment of radial wave equations (for the Langer modification, see Berry and Mount (1972), Morehead (1995)). It is not obvious to us what the Langer modification has to do with 1/2 that occurs in the present context, nor are we aware of any general rules about when in the asymptotics of angular momentum theory it is correct to replace a classical j by $j + 1/2$ (instead of $[j(j+1)]^{1/2}$ or something else). Schulten and Gordon (1975b) and Reinsch and Morehead (1999) obtain 1/2 as a part of their proper semiclassical analyses. Biedenharn and Louck (1981b) also speculate on the significance of 1/2. Roberts's (1999) derivation of the asymptotics of the 6j-symbols produces the 1/2's in some locations but not others. He shows that the answer is nevertheless valid to the same order in the asymptotic parameter as the result of Ponzano and Regge, but the error terms are different. We do not know which error term is smaller. One must be especially careful to include the 1/2's in the right places in the phase, as discussed by Biedenharn and Louck (1981b). According to Girelli and Livine (2005), different choices for the semiclassical replacement for the quantum number j have been made by various researchers in the field of quantum gravity. Here we have shown that the extra 1/2 is a necessary consequence of standard semiclassical theory. We remark in addition that with the inclusion of 1/2, the quantized spheres in angular momentum space are those with an area of $(2j^{\text{qu}} + 1)2\pi$, that is, they contain a number of Planck cells exactly equal to the dimension of the irrep, obviously a form of geometric quantization. In particular, the s-wave $j^{\text{qu}} = 0$ is represented by a sphere of nonzero radius, a case for which the replacement $j_r = j_r^{\text{qu}} + 1/2$ is declared by Biedenharn and Louck (1981b) to be 'clearly invalid'.

For the second basis contour on the jm-torus, the quantization condition is $2\pi(j_r + m_r) - 2(\pi/2) = 2\pi n_r'$, where n_r' is an integer. With (80), this implies $m_r = -j_r^{\text{qu}} + n_r'$. Combined with the classical restriction (40), this gives the usual range on magnetic quantum numbers. Again the semiclassical quantization is exact. In the case of J_{rz}, the eigenvalue of the operator is equal to the classical contour value m_r on the quantized torus (without any correction such as we see in (80)).

10.2. Quantizing the Wigner manifold

We begin the quantization of the Wigner manifold by guessing the basis contours of the fundamental group by inspection of figure 5. Taking the base (initial) point of the loops to be point a in the figure, we get three independent basis contours (call them C_1, C_2, C_3) by going around the 3-torus T_A in the three different directions. A fourth contour (call it C_4) is created by following an $SU(2)$ rotation about some axis by angle 2π, taking us along the path aba', which puts us half way around the torus T_A from the starting point, and then by applying half rotations along each of the three directions on the torus, taking us down along T_A in the diagram back to the starting point a. These four contours are not independent, since

$$2C_4 = C_1 + C_2 + C_3, \tag{81}$$

where addition of contours means concatenation, but they are convenient for studying the quantization conditions since a minimal set of three contours (not (C_1, C_2, C_3), but for example (C_1, C_2, C_4)) is less symmetrical. The fundamental group is $\mathbb{Z}^3$.

We may show the correctness of this guess by a topological argument. First, the Wigner manifold is the orbit of an $SU(2) \times T^3$ group action on the large phase space. Let (ψ_1, ψ_2, ψ_3) be coordinates on T^3, where $0 \leqslant \psi_r \leqslant 4\pi$. The action of $(u, (\psi_1, \psi_2, \psi_3)) \in SU(2) \times T^3$ on the large phase space is to multiply each spinor by u and then by $\exp(-\mathrm{i}\psi_r/2)$. The isotropy subgroup of this action consists of the identity $(1, (0, 0, 0))$ and the element $(-1, (2\pi, 2\pi, 2\pi))$, a normal subgroup isomorphic to $\mathbb{Z}_2$. Thus the Wigner manifold is diffeomorphic to $(SU(2) \times T^3)/\mathbb{Z}_2$, itself a group manifold, of which the original group $SU(2) \times T^3$ is a double cover. This cover is topologically simple since $SU(2)$ is simply connected (we can go to the universal cover if we wish by replacing T^3 by $\mathbb{R}^3$). Therefore the homotopy classes on the Wigner manifold are in one-to-one correspondence with classes of topologically inequivalent curves that go from the identity in $SU(2) \times T^3$ to one of the elements of the isotropy subgroup. Since $SU(2)$ is simply connected, such paths are characterized by choice of the end point (the element of the isotropy subgroup) and the winding numbers around the torus T^3. They are thus all 'linear combinations' of the four contours C_k, $k = 1, \ldots, 4$ defined above. Because of relation (81), however, the fundamental group is not $\mathbb{Z}^4$, but only $\mathbb{Z}^3$.

It is easy to compute the action along these contours. Contours C_r, $r = 1, 2, 3$, follow the flows of I_r's, and the action is the same as on the jm-torus, namely, $S_r = 4\pi j_r$. Along contour C_4 the spinor rotation makes no contribution to the action while the half rotation around the torus in all three angles gives the action

$$S_4 = 2\pi \sum_r j_r. \tag{82}$$

As for the Maslov indices, we compute the complex matrix of Poisson brackets whose rows are indexed by the functions $(I_1, I_2, I_3, J_x, J_y, J_z)$ and whose columns are indexed by $(\bar{z}_{11}, \bar{z}_{12}, \bar{z}_{21}, \bar{z}_{22}, \bar{z}_{31}, \bar{z}_{32})$. To within a multiplicative constant that we drop, the determinant is

$$\begin{vmatrix} \bar{z}_{11} & \bar{z}_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{z}_{21} & \bar{z}_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{z}_{31} & \bar{z}_{32} \\ \bar{z}_{12} & \bar{z}_{11} & \bar{z}_{22} & \bar{z}_{21} & \bar{z}_{32} & \bar{z}_{31} \\ -\bar{z}_{12} & \bar{z}_{11} & -\bar{z}_{22} & \bar{z}_{21} & -\bar{z}_{32} & \bar{z}_{31} \\ \bar{z}_{11} & -\bar{z}_{12} & \bar{z}_{21} & -\bar{z}_{22} & \bar{z}_{31} & -\bar{z}_{32} \end{vmatrix} = \text{const} \times \begin{vmatrix} \bar{z}_{11} & \bar{z}_{12} \\ \bar{z}_{21} & \bar{z}_{22} \end{vmatrix} \begin{vmatrix} \bar{z}_{21} & \bar{z}_{22} \\ \bar{z}_{31} & \bar{z}_{32} \end{vmatrix} \begin{vmatrix} \bar{z}_{31} & \bar{z}_{32} \\ \bar{z}_{11} & \bar{z}_{12} \end{vmatrix}. \tag{83}$$

The final product of determinants is interesting, since these are the $SU(2)$ invariants that Schwinger (Biedenharn and van Dam 1965) used to construct the rotationally invariant state $|j_1 j_2 j_3 \mathbf{0}\rangle$ (with $\bar{z}_{r\mu}$ replaced by $a^\dagger_{r\mu}$). Bargmann (1962) and Roberts (1999) make use of the same invariants. For our purposes we need the winding number of the loop traced out in the complex plane by the product of the three determinants as we follow the four basis contours on the Wigner manifold.

Proceeding as we did on the jm-torus, we find that the Maslov indices along the contours C_r, $r = 1, 2, 3$, are 4. For example, along the I_1-flow $\bar{z}_{11}$ and $\bar{z}_{12}$ get multiplied by $\exp(\mathrm{i}\psi_1/2)$, which causes two of the three determinants to be multiplied by the same factor, so the product gets multiplied by $\exp(\mathrm{i}\psi_1)$, which has winding number 2 and hence Maslov index 4 when ψ_1 goes from 0 to 4π. This is the same answer we found along the I_r-flows on the jm-torus; this was not exactly a foregone conclusion, even though the contours are the same, because the tangent planes are different. On the other hand, in both cases we find the result (80) for the eigenvalues $j_r^{\rm qu}$ of the operators $\hat{I}_r$, which of course must not depend on how we compute them. As for contour C_4 the first leg, a rotation by angle 2π about some axis, leaves all three 2×2 determinants in (83) invariant, so the big determinant in the complex plane does not move. As for the second leg, since each ψ_r only goes from 0 to 2π we get a winding number of 1 along each I_r-flow, but since there are three of them the total winding number is 3 and Maslov index is 6.

Combining this result with (82), we obtain the Bohr–Sommerfeld quantization condition for contour C_4 in the form

$$2\pi \sum_r j_r - 6\frac{\pi}{2} = 2\pi \times \text{ integer}, \tag{84}$$

or, with (80),

$$\sum_r j_r^{\rm qu} = \text{integer}. \tag{85}$$

This is precisely the condition that the three quantum angular momenta must satisfy, in addition to the triangle inequalities, that they may add up to zero. It emerges in a semiclassical analysis because the Wigner manifold is not quantized otherwise.

In conclusion, the Bohr–Sommerfeld quantization conditions applied to the jm-torus and the Wigner manifold give us a complete (and exact) accounting of all the quantum numbers and the restrictions on them that appear in the coupling of three angular momenta with a resultant of zero. It also allows us to identify the classical manifold (that is, its contour values) with a given set of quantum numbers.

11. The amplitude determinant

The generic semiclassical eigenfunction of a complete set of commuting observables is given by (1), with the amplitude determinant expressed in terms of Poisson brackets by (2). These formulae apply in particular to the state $|j_1 j_2 j_3 m_1 m_2 m_3\rangle$ on the left of the matrix element (22), which is supported by the jm-torus in the large phase space. The state on the right, $|j_1 j_2 j_3 \mathbf{0}\rangle$, however, which is supported by the Wigner manifold, is an eigenfunction of observables that do not commute. Therefore we must rethink the derivation of equations (1) and (2) to see what changes in this case. In particular, we must see what happens to the Poisson bracket expression for the amplitude determinant, which is the solution of the simultaneous amplitude transport equations for the collection of observables. As it turns out, nothing changes, the wavefunction is still given by equations (1) and (2), with the (now noncommuting) observables used in the amplitude determinant. In addition, there is a certain understanding about how the volume V in (1) is computed, since the angles α conjugate to A's are no longer meaningful.

Once this is done, we must evaluate the scalar product of the two wavefunctions by stationary phase. If both states were eigenstates of complete sets of commuting observables, then the answer would be (5) with amplitude determinant (4), but again we must rethink the derivation of this result since the observables for one of the wavefunctions do not commute. Again, the answer turns out to be given by formulae (5) and (4) of section 2, with a proper understanding of the meanings of the volume factors.

Having established these facts, we can then proceed to the (easy) calculation of the amplitude determinant for the $3j$-symbol in terms of Poisson brackets, and finally put the remaining pieces together to get the leading asymptotic form of the $3j$-symbols.

11.1. Amplitude determinant for noncommuting observables

We begin showing that equations (1) and (2) are valid for the state $|j_1 j_2 j_3 \mathbf{0}\rangle$, with a proper definition of the volume factors. The classical functions defining the Wigner manifold are

$(I_1, I_2, I_3, J_x, J_y, J_z)$. Let us refer to these collectively as $A_k, k = 1, \ldots, 6$, let us write $x^i, i = 1, \ldots, 6$, for the configuration space coordinates instead of the notation $(x_{11}, \ldots, x_{32})$ used above, and let us adopt the summation convention. The functions A_k form a Lie algebra, that is, $\{A_k, A_l\} = c^m_{kl} A_m$, where c^m_{kl} are the structure constants. The Wigner manifold is a compact group manifold with this Lie algebra, on which the Haar measure is both left- and right-invariant. This density is also invariant under the flows generated by the right-invariant vector fields, which in our case are the Hamiltonian flows of the functions A_i. The projection of this density onto configuration space is the density that provides the solution of the simultaneous amplitude transport equations for the functions A_i. These are the basic geometrical facts, which we now present more explicitly in coordinate language.

The amplitude transport equations for the functions $A_k, k = 1, \ldots, 6$, are

$$\frac{\partial}{\partial x^i}\left[\Omega(x)\frac{\partial A_k}{\partial p_i}\right] = 0, \tag{86}$$

where p_i are the momenta conjugate to x^i. These are six simultaneous equations that must be solved for the density $\Omega(x)$ on configuration space. Note that $\partial A_k/\partial p_i = \{x^i, A_k\} = \dot{x}^i_{(k)}$, the latter being notation we shall use for the velocity in configuration space along the Hamiltonian flow generated by A_k. The amplitude transport equation is a continuity equation, which is form-invariant under general coordinate transformations.

Let us pick one of the branches of the inverse projection from configuration space onto the Lagrangian (Wigner) manifold. We shall suppress the branch index in the following. Let u^i, $i = 1, \ldots, 6$, be an arbitrary set of local coordinates on the Wigner manifold, which we extend in a smooth but arbitrary manner into some small neighbourhood of the Wigner manifold, so that partial derivatives of u^i with respect to all phase space coordinates are defined. Assuming we are not at a caustic, the transformation from x^i to u^i is locally one-to-one, and the Jacobian $\partial u^i/\partial x^j$ is nonsingular. Under the inverse projection or coordinate transformation $x \to u$, the flow velocity transforms according to

$$\dot{u}^i_{(k)} = \frac{\partial u^i}{\partial x^j}\dot{x}^j_{(k)} = \{u^i, A_k\} = X^i_{(k)}, \tag{87}$$

which defines the quantities $X^i_{(k)}$. As a matrix, $X^i_{(k)}$ is nonsingular because the flow vectors are linearly independent of the Wigner manifold. As for the density, it transforms according to

$$\sigma(u) = \Omega(x)\left|\det\frac{\partial x^l}{\partial u^m}\right|, \tag{88}$$

so that the amplitude transport equations, lifted to the Wigner manifold, become

$$\frac{\partial}{\partial u^i}\left[\sigma(x)X^i_{(k)}\right] = 0. \tag{89}$$

Now define $\Lambda^{(k)}_j$ as the matrix inverse to $X^i_{(k)}$,

$$\Lambda^{(k)}_i X^i_{(l)} = \delta^k_l. \tag{90}$$

As we will prove momentarily, the solution of equations (89) is

$$\sigma(u) = \left|\det \Lambda^{(k)}_j\right|, \tag{91}$$

which, by (88), gives us the solution of (86),

$$\begin{aligned}\Omega(x) &= \left|\det_{kl}\left(\Lambda^{(k)}_i\frac{\partial u^i}{\partial x^l}\right)\right| = \left|\det_{kl}\left(X^i_{(k)}\frac{\partial x^l}{\partial u^i}\right)\right|^{-1} \\ &= \left|\det_{kl}\left(\dot{u}^i_{(k)}\frac{\partial x^l}{\partial u^i}\right)\right|^{-1} = \left|\det_{kl}\dot{x}^l_{(k)}\right| = \left|\det_{kl}\{x^l, A_k\}\right|^{-1}.\end{aligned} \tag{92}$$

In carrying out these manipulations it is important to note that $\partial u^i/\partial x^j$ is taken at constant A_k not p_k. Thus the amplitude determinant for the wavefunction $\psi(x)$ associated with the Wigner manifold has the same Poisson bracket form shown in (2), that is, in spite of the fact that A_i do not commute.

The essential differential geometry of these manipulations is that $X_{(k)} = X^i_{(k)}\partial/\partial u^i$ are the Hamiltonian vector fields on the Wigner manifold associated with functions A_k, $\lambda^{(k)} = \Lambda^{(k)}_i\,\mathrm{d}u^i$ are the dual forms, $\lambda^{(k)}X_{(l)} = \delta^k_l$, and $\sigma = \lambda^1 \wedge \cdots \wedge \lambda^6 = \sigma(u)\,\mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^6$ is the Haar measure. Condition (89) is equivalent to $L_{X_{(k)}}\sigma = 0$.

To prove (91) in coordinates we substitute it into (89) and expand out the derivative, obtaining an expression proportional to

$$X^i_{(k),i} - X^i_{(k)}X^l_{(m),i}\Lambda^{(m)}_l \tag{93}$$

using commas for derivatives. Then we use the Lie bracket of the vector fields $X_{(k)}$,

$$[X_{(k)}, X_{(m)}]^l = X^i_{(k)}X^l_{(m),i} - X^i_{(m)}X^l_{(k),i} = -c^n_{km}X^l_{(n)}, \tag{94}$$

where c^n_{km} are the structure constants. Here we use the identity expressing the Lie bracket of Hamiltonian vector fields for two functions in terms of the Hamiltonian vector field of their Poisson bracket, $[X_H, X_K] = -X_{\{H,K\}}$ (Arnold 1989). Thus (93) becomes simply c^m_{km}, which vanishes since for the group in question the structure constants are completely antisymmetric.

Finally to normalize the semiclassical eigenfunction supported by the Wigner manifold we use the stationary phase approximation to compute the integral

$$\int \mathrm{d}x \left|\sum_{\mathrm{br}} \Omega(x)^{1/2}\exp[\mathrm{i}S(x) - \mathrm{i}\mu\pi/2]\right|^2, \tag{95}$$

where the sum is over branches and the branch index is suppressed. Cross terms do not contribute, and when the integral is lifted to the Wigner manifold it just gives the volume of that manifold with respect to the Haar measure,

$$\sum_{\mathrm{br}}\int \frac{\mathrm{d}x}{|\det\{x^i, A_j\}|} = \int \mathrm{d}u\,\sigma(u) = V_W, \tag{96}$$

where V_W is given by (46).

11.2. Matrix elements for noncommuting observables

Now we write the $3j$ matrix element (22) as $\langle b|a\rangle$, where $A = (I_1, I_2, I_3, J_x, J_y, J_z)$ and $B = (I_1, I_2, I_3, J_{1z}, J_{2z}, J_{3z})$. Actually this is not the most convenient form, since J_z in the A-list is a function of J_{rz} in the B-list. We fix this by performing a canonical transformation $(\phi_1, \phi_2, \phi_3, J_{1z}, J_{2z}, J_{3z}) \to (\tilde\phi_1, \tilde\phi_2, \tilde\phi_3, \tilde J_{1z}, \tilde J_{2z}, J_z)$ on the functions in the B-list, generated by

$$F_2(\phi_1, \phi_2, \phi_3, \tilde J_{1z}, \tilde J_{2z}, J_z) = \phi_1\tilde J_{1z} + \phi_2\tilde J_{2z} + \phi_3(J_z - \tilde J_{1z} - \tilde J_{2z}). \tag{97}$$

This gives $\tilde J_{1z} = J_{1z}$, $\tilde J_{2z} = J_{2z}$, $J_z = J_{1z}+J_{2z}+J_{3z}$ and $\tilde\phi_1 = \phi_1-\phi_3$, $\tilde\phi_2 = \phi_2-\phi_3$, $\tilde\phi_3 = \phi_3$. The linear transformation in the angles has unit determinant, so the volume of the jm-torus is still given by (43). Dropping the tildes, the B-list is now $(I_1, I_2, I_3, J_{1z}, J_{2z}, J_z)$, which has four functions in common with the A-list.

Now the integral we must evaluate is

$$\langle b|a\rangle = \frac{1}{\sqrt{V_A V_B}}\int \mathrm{d}x\,\frac{1}{|\det\{x^i, A_j\}|^{1/2}}\,\frac{1}{|\det\{x^i, B_j\}|^{1/2}}\sum_{\mathrm{br}}\exp\{\mathrm{i}[S_A(x) - S_B(x) - \mu\pi/2]\}, \tag{98}$$

where the sum is over all branches of the projections of the two manifolds, and where μ just stands for whatever Maslov index appears in a given term (different μ's are not necessarily equal). An integral like this was evaluated by Littlejohn (1990), using the angles conjugate to A's and B's, but those do not all exist in the present circumstances and we must evaluate the integral in a different way.

Let us write $A = (C, D)$ and $B = (C, E)$, where $C = (I_1, I_2, I_3, J_z)$ are the four observables in common in the A- and B-lists, and where $D = (J_x, J_y)$ and $E = (J_{1z}, J_{2z})$ are the two pairs of observables that are distinct. The stationary phase set of the integral (98) consists of points x where $\partial(S_A - S_B)/\partial x^i = 0$, that is, it is the projection onto configuration space of the intersection of the A-manifold and the B-manifold. That intersection, which we denote by I, was studied in section 8 (it is a 4-torus). It is the simultaneous level set of all of A's and B's, and at the same time the orbit of the commuting Hamiltonian flows generated by C's. Its projection onto configuration space is a four-dimensional region.

We introduce a local coordinate transformation in configuration space $x \to (y, z)$ where the four y's are coordinates along the stationary phase set and the two z's are transverse to it. We let the stationary phase set itself be specified by $z = 0$. We let (u, v) be the momenta conjugate to (y, z). Then the two amplitude determinants in (98) may be combined with the Jacobian of the coordinate transformation to result in the square root of the product of two determinants, one of which is

$$\det \frac{\partial(y,z)}{\partial x} \det\{x, A\} = \det \begin{pmatrix} \{y, C\} & \{y, D\} \\ \{z, C\} & \{z, D\} \end{pmatrix}, \tag{99}$$

and the other of which is the same but with the substitutions $A \to B$, $D \to E$. But since C's generate flows along I, we have $\{z^i, C_j\} = 0$, and the lower left block of the two matrices vanishes. Thus, the product of the two determinants becomes

$$[\det\{y, C\}]^2 \det\{z, D\} \det\{z, E\}, \tag{100}$$

the square root of which appears in the denominator of the integrand. Evaluating the final two Poisson brackets in the $(y, z; u, v)$ canonical coordinates, we have

$$\{z^i, D_j\} = \frac{\partial D_j}{\partial v_i}, \qquad \{z^i, E_j\} = \frac{\partial E_j}{\partial v_i}. \tag{101}$$

We perform the z-integration by stationary phase, expanding S_A and S_B, regarded as functions of (y, z), to second order in z for a fixed value of y, and simply evaluating the amplitude at $z = 0$ (that is, on I). To within a phase, the z-integration gives

$$2\pi \left| \det \left(\frac{\partial^2 S_A}{\partial z \partial z} - \frac{\partial^2 S_B}{\partial z \partial z} \right) \right|^{-1/2}. \tag{102}$$

The determinant in this result must be multiplied by the determinants of the matrices (101) to get the overall determinant in the denominator after the z-integration. The product of these three determinants is the determinant of the matrix

$$\frac{\partial D}{\partial v} \left[\left(\frac{\partial v}{\partial z} \right)_{yCD} - \left(\frac{\partial v}{\partial z} \right)_{yCE} \right] \left(\frac{\partial E}{\partial v} \right)^T, \tag{103}$$

where it is understood that a partial derivative stands for a matrix whose row index is given by the numerator and column index by the denominator, unless the matrix transpose or inverse is indicated, in which case the rule is reversed. Also, if a partial derivative is shown without subscripts, then it is assumed that it is computed in the canonical coordinates $(y, z; u, v)$, and otherwise the variables to be held fixed are explicitly indicated. In the two middle matrices in (103), the variables held fixed amount to differentiating v with respect to z along the A- and

B-manifolds, respectively, since $\partial S_A/\partial z = v(x, A)$ and $\partial S_B/\partial z = v(x, B)$. Note that these two matrices are symmetric.

Now we express the two matrices in the middle of (103) purely in terms of partial derivatives computed in the canonical $(y, z; u, v)$ coordinates. We do this by writing out the Jacobian matrix $\partial(y, z; C, D)/\partial(y, z; u, v)$ and the inverse Jacobian $\partial(y, z; u, v)/\partial(y, z; C, D)$, multiplying the two together to obtain a series of identities connecting the forward and inverse Jacobian blocks, and then solving for the inverse Jacobian blocks in terms of the forward ones. We note that the block $\partial C/\partial v$ of the forward Jacobian vanishes, since it is $\{z, C\}$. Thus we find

$$\begin{aligned} \left(\frac{\partial v}{\partial z}\right)_{yCD} &= \left(\frac{\partial D}{\partial v}\right)^{-1}\left[\frac{\partial D}{\partial u}\left(\frac{\partial C}{\partial u}\right)^{-1}\frac{\partial C}{\partial z} - \frac{\partial D}{\partial z}\right], \\ \left(\frac{\partial v}{\partial z}\right)_{yCE} &= \left[\left(\frac{\partial C}{\partial z}\right)^{T}\left(\frac{\partial C}{\partial u}\right)^{-1T}\left(\frac{\partial E}{\partial u}\right)^{T} - \left(\frac{\partial E}{\partial z}\right)^{T}\right]\left(\frac{\partial E}{\partial v}\right)^{-1T}. \end{aligned} \tag{104}$$

Upon substituting these into (103), that matrix becomes

$$\begin{aligned} &\frac{\partial D}{\partial v}\left(\frac{\partial E}{\partial z}\right)^{T} - \frac{\partial D}{\partial z}\left(\frac{\partial E}{\partial v}\right)^{T} + \frac{\partial D}{\partial u}\left(\frac{\partial C}{\partial u}\right)^{-1}\frac{\partial C}{\partial z}\left(\frac{\partial E}{\partial v}\right)^{T} \\ &\qquad - \frac{\partial D}{\partial v}\left(\frac{\partial C}{\partial z}\right)^{T}\left(\frac{\partial C}{\partial u}\right)^{-1T}\left(\frac{\partial E}{\partial u}\right)^{T}, \end{aligned} \tag{105}$$

where the first two terms are the beginning of the Poisson bracket $\{E, D\}$. As for the last two terms, we write out the vanishing Poisson brackets $\{C, E\}$ and $\{C, D\}$ in the $(y, z; u, v)$ coordinates, making use of $\partial C/\partial v = 0$, to obtain

$$\begin{aligned} \frac{\partial C}{\partial z}\left(\frac{\partial E}{\partial v}\right)^{T} &= \frac{\partial C}{\partial u}\left(\frac{\partial E}{\partial y}\right)^{T} - \frac{\partial C}{\partial y}\left(\frac{\partial E}{\partial u}\right)^{T}, \\ \frac{\partial D}{\partial v}\left(\frac{\partial C}{\partial z}\right)^{T} &= \frac{\partial D}{\partial y}\left(\frac{\partial C}{\partial u}\right)^{T} - \frac{\partial D}{\partial u}\left(\frac{\partial C}{\partial y}\right)^{T}. \end{aligned} \tag{106}$$

Actually the matrix of Poisson brackets $\{C, D\}$ does not vanish everywhere in phase space, just on the A- (or Wigner) manifold, and in particular on the intersection I which is where we are evaluating them. Now substituting equations (106) into the last two terms of (105), those terms become

$$\frac{\partial D}{\partial u}\left(\frac{\partial E}{\partial y}\right)^{T} - \frac{\partial D}{\partial y}\left(\frac{\partial E}{\partial u}\right)^{T} + \frac{\partial D}{\partial u}\left[\left(\frac{\partial C}{\partial y}\right)^{T}\left(\frac{\partial C}{\partial u}\right)^{-1T} - \left(\frac{\partial C}{\partial u}\right)^{-1}\frac{\partial C}{\partial y}\right]\left(\frac{\partial E}{\partial u}\right)^{T}, \tag{107}$$

in which the first two terms give us the remainder of the Poisson bracket $\{E, D\}$. As for the last major term, the factor in the square brackets vanishes, as we see by writing out the vanishing Poisson bracket $\{C, C\}$ in coordinates $(y, z; u, v)$ and using $\partial C/\partial v = 0$.

As a result the integral (98) becomes

$$\langle b|a\rangle = \frac{2\pi}{\sqrt{V_A V_B}} \sum_{\text{br}} \int \frac{\mathrm{d}y}{|\det\{y, C\}|} \frac{\mathrm{e}^{\mathrm{i}(S_I - \mu\pi/2)}}{|\det\{E, D\}|^{1/2}}, \tag{108}$$

where the branch sum runs over all branches of the projection of I onto configuration space as well as the two disconnected components of I (the two 4-tori discussed in section 8), and where S_I is the phase on a given connected component of I (this is the phase $\pm S_{jm}$ computed in section 9). We have also dropped an overall phase, and we are not attempting to compute the Maslov indices in detail. The amplitude determinant has been reduced to a 2×2 matrix of Poisson brackets of the observables in the A- and B-lists that differ exactly as in (4). Calculating this matrix explicitly, we find

$$|\det\{E, D\}| = \begin{vmatrix} \{J_{1z}, J_x\} & \{J_{1z}, J_y\} \\ \{J_{2z}, J_x\} & \{J_{2z}, J_y\} \end{vmatrix} = \begin{vmatrix} J_{y1} & -J_{x1} \\ J_{y2} & -J_{x2} \end{vmatrix}$$
$$= |J_{x1}J_{y2} - J_{x2}J_{y1}| = |\mathbf{z} \cdot (\mathbf{J}_1 \times \mathbf{J}_2)| = 2\Delta_z, \qquad (109)$$

where Δ_z is the projection of the area of the triangle Δ onto the x–y plane (see equation (50) and figure 2 of Ponzano and Regge (1968)). This quantity is invariant under rotations about the z-axis, that is, it Poisson commutes with J_z. It also Poisson commutes with the other three variables in the C-list, (I_1, I_2, I_3), and so is constant on the intersection I and can be taken out of the y-integral. The same applies to the phase factor, since S_I is also constant on the I-manifold. Then the y-integral can be done, since $|\det\{y, C\}|$ is just the Jacobian connecting y with the angle variables conjugate to $C = (I_1, I_2, I_3, J_z)$, denoted above by $(\psi_1, \psi_2, \psi_3, \phi)$. Thus the y-integral just gives the volume V_I of the intersection I with respect to these angles, see (56). In fact, had the variables C not been commuting, but if they had formed a Lie algebra, then V_I would be the volume of I with respect to the Haar measure of the corresponding group. This circumstance arises, for example, in a similar treatment of the $6j$-symbol.

As a result of these rather lengthy manipulations of amplitude determinants, we obtain the final, simple result,

$$\langle b|a\rangle = \frac{2\pi}{\sqrt{V_A V_B}} \sum_{\text{br}} V_I \frac{\mathrm{e}^{\mathrm{i}(S_I - \mu\pi/2)}}{|\det\{E, D\}|^{1/2}}, \qquad (110)$$

where the branches now run over just the two disconnected pieces of the intersection I. This is a version of (5), with the right understanding of the volume measures, generalized to the case at hand in which the observables do not commute. The actual calculation of the final amplitude determinant takes just one line, equation (109).

In fact, for our application the volume V_I and the remaining amplitude determinant are the same for both branches and can be taken out of the sum. The relative Maslov index between the two branches is 1; we will not belabour this point since the answer is already known. We simply note that by splitting the Maslov phase $\mathrm{i}\pi/2$ between the two branches and substituting $V_A = V_W$, $V_B = V_{jm}$, we obtain to within an overall phase the result of Ponzano and Regge,

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (\text{phase}) \times \frac{\cos(S_{jm} + \pi/4)}{\sqrt{2\pi\,\Delta_z}}. \qquad (111)$$

12. Conclusions

In many ways the $3j$-symbol is not as interesting as the $6j$-symbol, of which it is a limiting case. We intended our work on the $3j$-symbol as a warm-up exercise, expecting a routine application of semiclassical methods for integrable systems. The nongeneric Lagrangian (Wigner) manifold was a surprise. Similar nongeneric Lagrangian manifolds occur also in the semiclassical analysis of the $6j$- and $9j$-symbols.

If all one wants is a derivation of an asymptotic formula, then there are many ways to proceed. For example, one can simply take the expression for the symbol due to Wigner ($3j$) or Racah ($6j$) as a sum over a single index, and apply standard asymptotic methods (Stirling's approximation, Poisson sum rule, etc). But if one wants a derivation that reveals the geometrical meaning of the classical objects that emerge (the triangle, the tetrahedron, etc), then an approach such as ours may be preferable.

Our approach is more geometrical than earlier ones, and in that respect is closer in spirit to the work of Roberts (1999), Freidel and Louapre (2003) and later authors. It is likely that at some deeper level all these methods are the same, although superficially we see only a little similarity between our work and these others.

One may also desire a method that makes the symmetries of the symbol manifest. Our analysis does not do this for the 3*j*-symbol, but those symmetries are not manifest in Wigner's definition of the 3*j*-symbol that we employ as our starting point, either. To bring the symmetries out it seems necessary to employ some construction related to Schwinger's generating functions, which involve lifting the definitions into higher dimensional spaces.

Our method of calculating amplitude determinants in terms of Poisson brackets may have computational advantages in other applications as well. The method can be remarkably easy to use. For example, the 6*j*-symbol can be defined as a matrix element,

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_4 & j_3 & j_{23} \end{Bmatrix} = \text{const} \times \langle j_1 j_2 j_3 j_4 j_{23} \mathbf{0} | j_1 j_2 j_3 j_4 j_{12} \mathbf{0} \rangle, \tag{112}$$

which is the unitary matrix in (j_{12}, j_{23}) defining a change of basis in the subspace in which four angular momenta of given lengths add up to zero ($\mathbf{0}$ means $\mathbf{J} = \mathbf{0}$). In this case there are eight observables on each side of the matrix element, of which seven are common and one is different. Thus the amplitude of the 6*j*-symbol is the inverse square root of the single Poisson bracket,

$$\{\mathbf{J}_{23}^2, \mathbf{J}_{12}^2\} = 4\mathbf{J}_1 \cdot (\mathbf{J}_2 \times \mathbf{J}_3), \tag{113}$$

as follows immediately from (37). One sees immediately that it is proportional to the volume of the tetrahedron. A similarly easy calculation is possible for the 9*j*-symbol. It is harder, however, to express these amplitudes in terms of the quantum numbers (the magnitudes j_r), that is, to translate these magnitudes into vectors $\mathbf{J}_r$ that lie on the stationary phase set. We shall report on these and other extensions of our work in future publications.

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